

研究ノート

TRANSFORMATIONS OF HEEGAARD DIAGRAMS CORRESPONDING TO THOSE OF HEEGAARD CUT DIAGRAMS II¹

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1. Introduction

This paper is continued from the last number [10]. We use the same notations, definitions as in [10].

Theorem 3². *Let $U \cup V$ be a Heegaard splitting of M^3 . U, V is a Heegaard-handlebody, respectively. Let $(U; m, l) \cup (V; l, m)$ be Heegaard diagrams associated with $U \cup V$. $\{m, l\}$ is a meridian-longitude system of $(U; m, l)$.*

(7) *Let the following figure $U6-A$ be a part of $(U; m, l)$. The longitude l_i crosses the meridians $\{m_1, \dots, m_g\}$. Let $U6-B$ be $g-3$ handles added to $U6-A$. l_i is decomposed into $g-2$ circles $\{\tilde{l}_1, \dots, \tilde{l}_{g-2}\}$ and each circle goes around on three handles. Then there exists a transformation from $U6-A$ into $U6-B$ of $(U; m, l)$.*

(7') *Let $V6-A'$ be a part of $(V; l, m)$. $V6-A'$ is the dual part of $U6-A$, i.e., each longitude m_i ($i = 1, \dots, g$) goes around on the handle h_i' . We cut off V at the meridian disk D_i' ($\partial D_i' = l_i$) and put $g-2$ handles over the two cutting places as shown in $V6-B'$. Then there exists a transformation from $V6-A'$ into $V6-B'$ of $(V; l, m)$ so that m_i goes around on the new handle.*

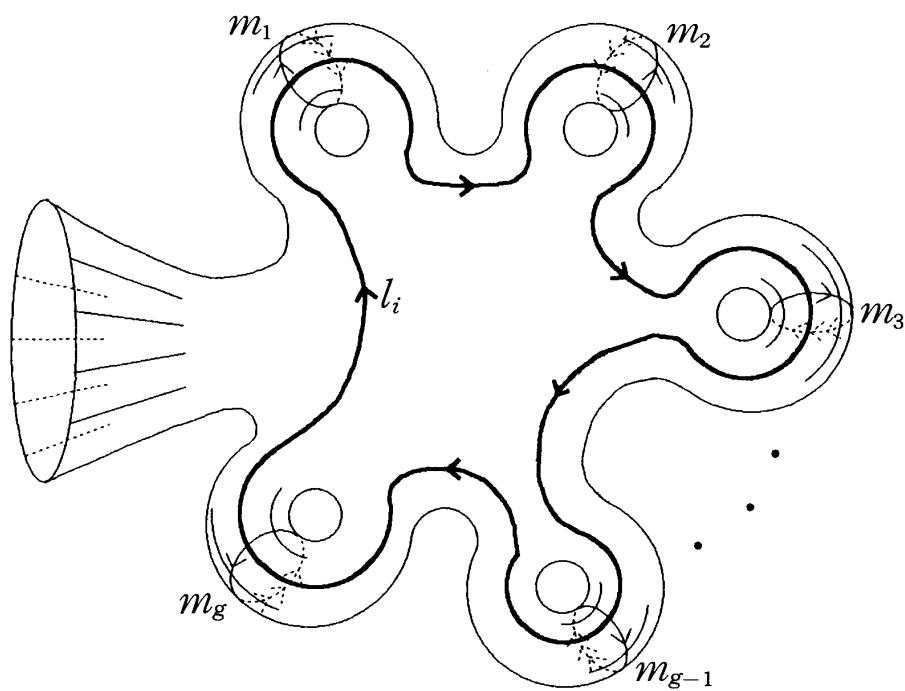
(8) *Let the following figure $U7-A$ be a part of $(U; m, l)$. The longitude l_i goes around on two handles and l_j goes around on another two handles. Let $U7-B$ be a handle added to $U7-A$. Then there exists a transformation from $U7-A$ into $U7-B$ of $(U; m, l)$ so that l_i and l_j go around on the handle.*

(8') *Let $V7-A'$ be a part of $(V; l, m)$. $V7-A'$ is the dual part of $U7-A$. Let $V7-B'$ be a handle added to $V7-A'$. Then there exists a transformation from $V7-A'$ into $V7-B'$ of $(V; l, m)$ so that the longitude \tilde{m} crosses the meridians l_i, l_j and \tilde{l} .*

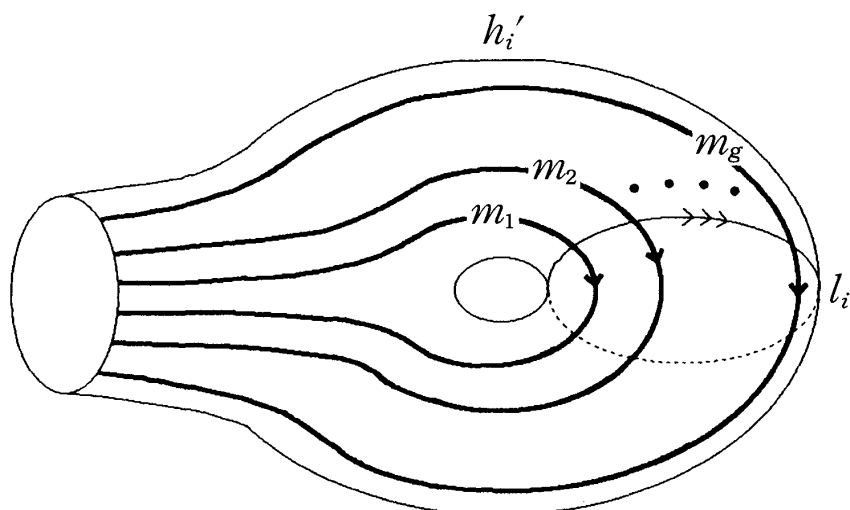
¹ Partially supported by the Mathematical Economics Study Group of Niigata Sangyou University.

² We use Theorem 3 as a serial notation.

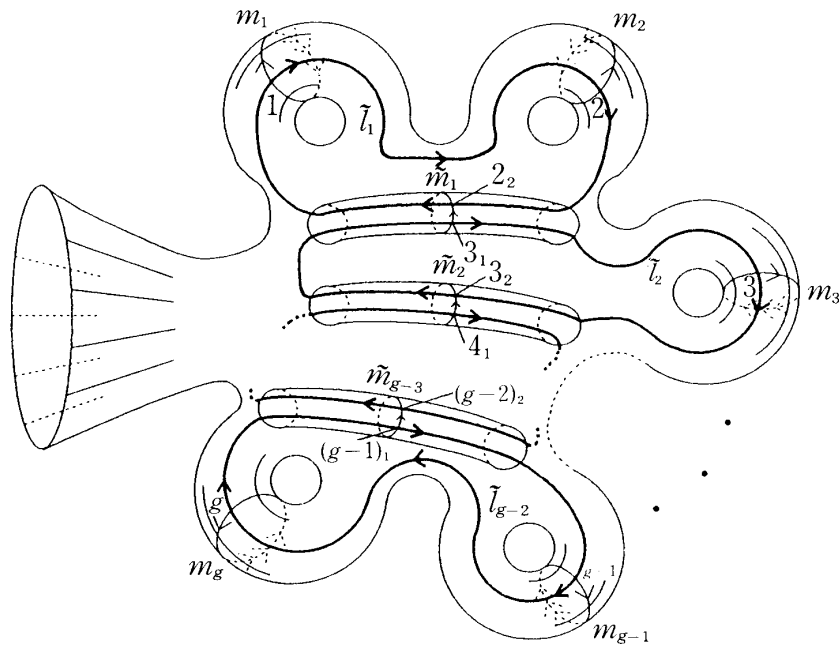
³ M^3 denotes a connected orientable closed 3-manifold.



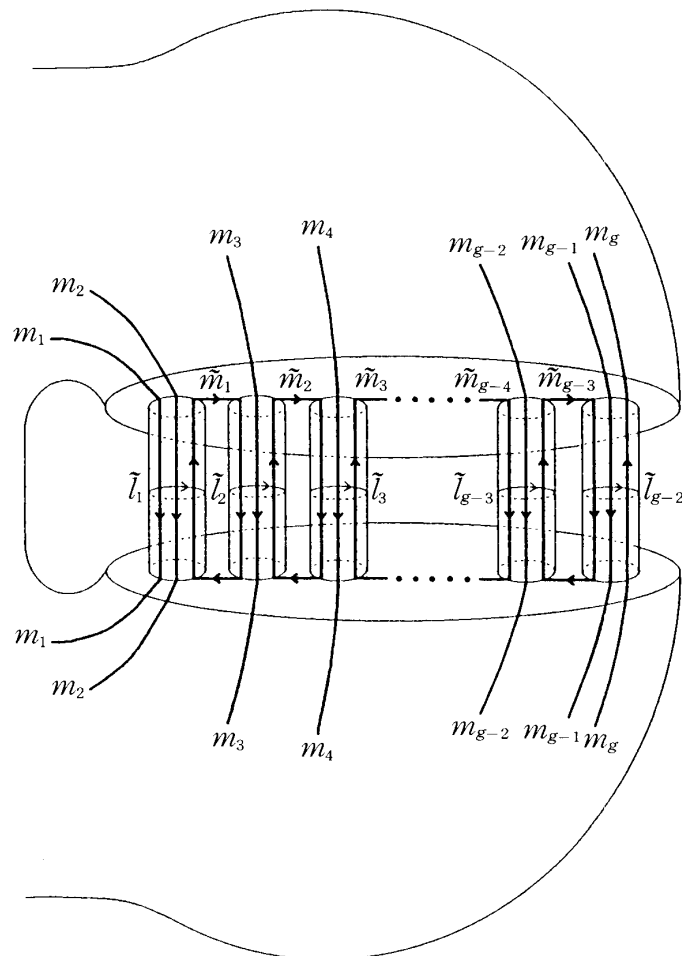
U6-A



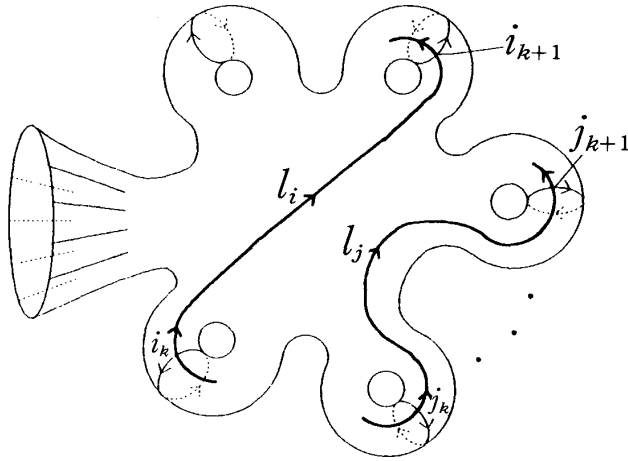
V6-A'



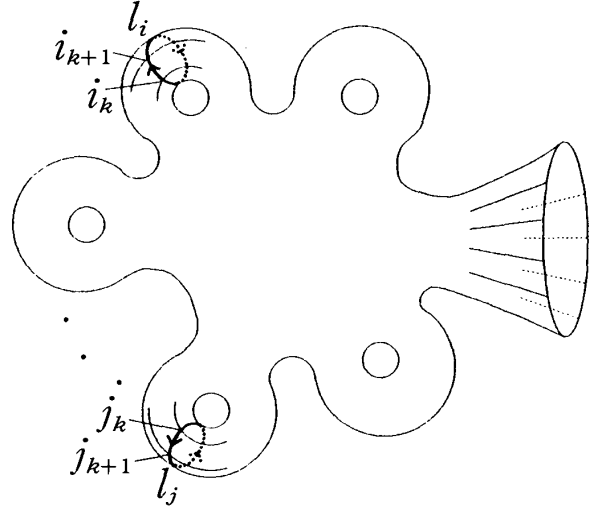
U6-B



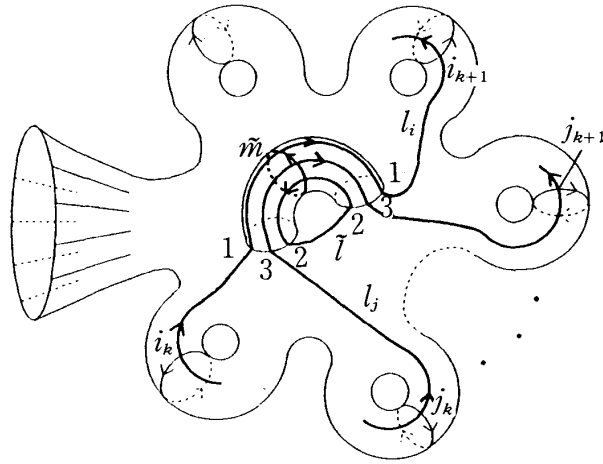
V6-B'



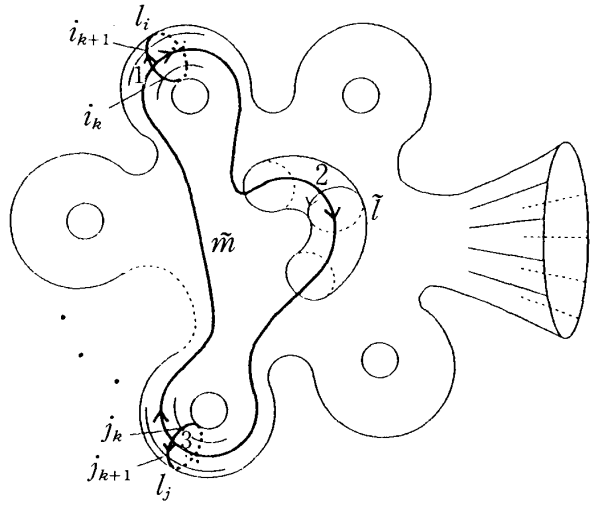
U7-A



V7-A'



U7-B



V7-B'

2. Connectedness of Heegaard diagrams and Heegaard cut diagrams

Let $(U; m, l) \cup (V; l, m)$ be genus n (≥ 2) H-diagrams of (U, V, F) of M^3 . $m = \{m_1, \dots, m_n\}$ and $l = \{l_1, \dots, l_n\}$. Let $G(m, l) \cup G(l, m)$ be the H-cut-diagrams of $(U; m, l) \cup (V; l, m)$, respectively. Let $G(m, l) \cup G(l, m)$ be the same presentation as in Def. 7 in [10, p. 45]. We choose $n-1$ edges from $\{j_k(l_j)j_{k+1}\}$ in $\{l_j\}(j = 1, \dots, n)$ of $G(m, l)$ and let these edges be $\{L_1, \dots, L_{n-1}\}$.

Definition 11. $(U; m, l)$ is called *connected* for the meridian system m if $\{L_1, \dots, L_{n-1}\}$ can be chosen so that $(L_1 \cup \dots \cup L_{n-1}) \cup (m_1 \cup \dots \cup m_n)$ becomes a connected graph. If $(U; m, l)$ does not connect, then it is called *disconnected* for m . The closed connected orientable 3-manifolds, which have genus 1 H-diagram are 3-sphere S^3 , lens spaces $L(p, q)$ and $S^2 \times S^1$. p and q of $L(p, q)$ are integers and relatively prime. We define that each genus 1 H-diagram of S^3 and $L(p, q)$ is connected and that of $S^2 \times S^1$ is disconnected for $m = \{m_1\}$.

Definition 12. If the closures of connected components of $S^2 - |G(m, l)|$ consist of 2-disks, then $G(m, l)$ is called *connected for m and l* . If $G(m, l)$ is not connected, then it is called *disconnected for m and l* .

Fig. 1 shows a disconnected H-cut-diagram $G(m, l)$ of $(U; m, l)$ of $S^2 \times S^1$ but here the connection of H-diagram $(U; m, l)$ is made.

Therefore, we see the following obvious proposition.

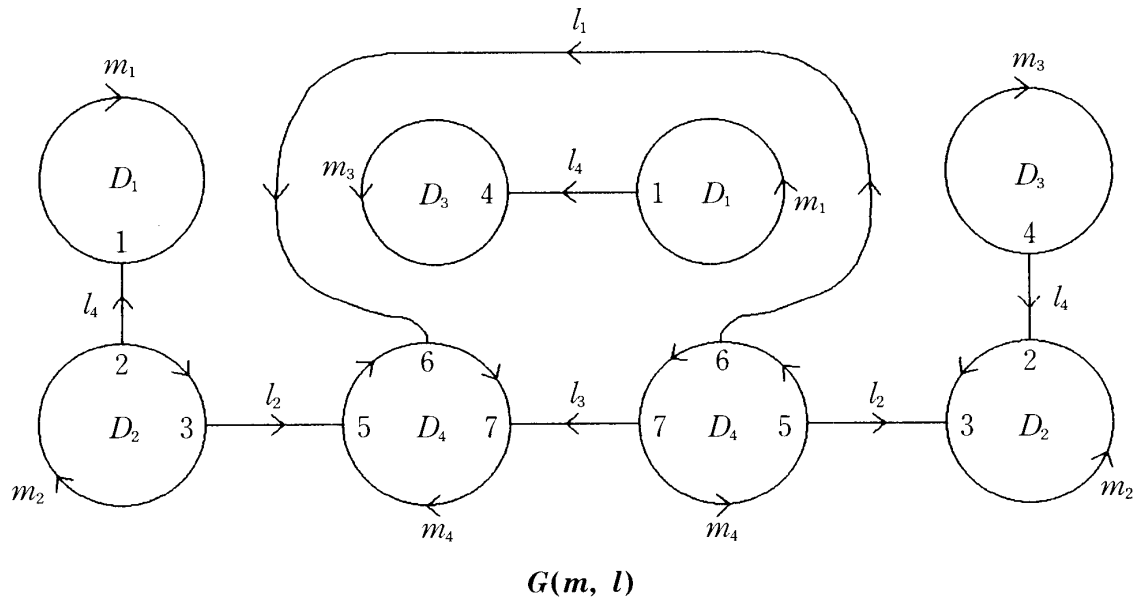


Fig. 1

Proposition 2. The connectedness of H-cut-diagram $G(m, l)$ is equivalent to that of $G(l, m)$. If $G(m, l)$ of H-diagram $(U; m, l)$ is connected, then $(U; m, l)$ also becomes connected but the reverse of this does not hold generally. If H-genus = 1, then the connectedness of $(U; m, l)$ is equivalent to that of $G(m, l)$ where $m = \{m_1\}$ and $l = \{l_1\}$.

3. Transformations of Heegaard cut diagrams

From now on, we give transformations of H-cut-diagrams besides those given in [10].

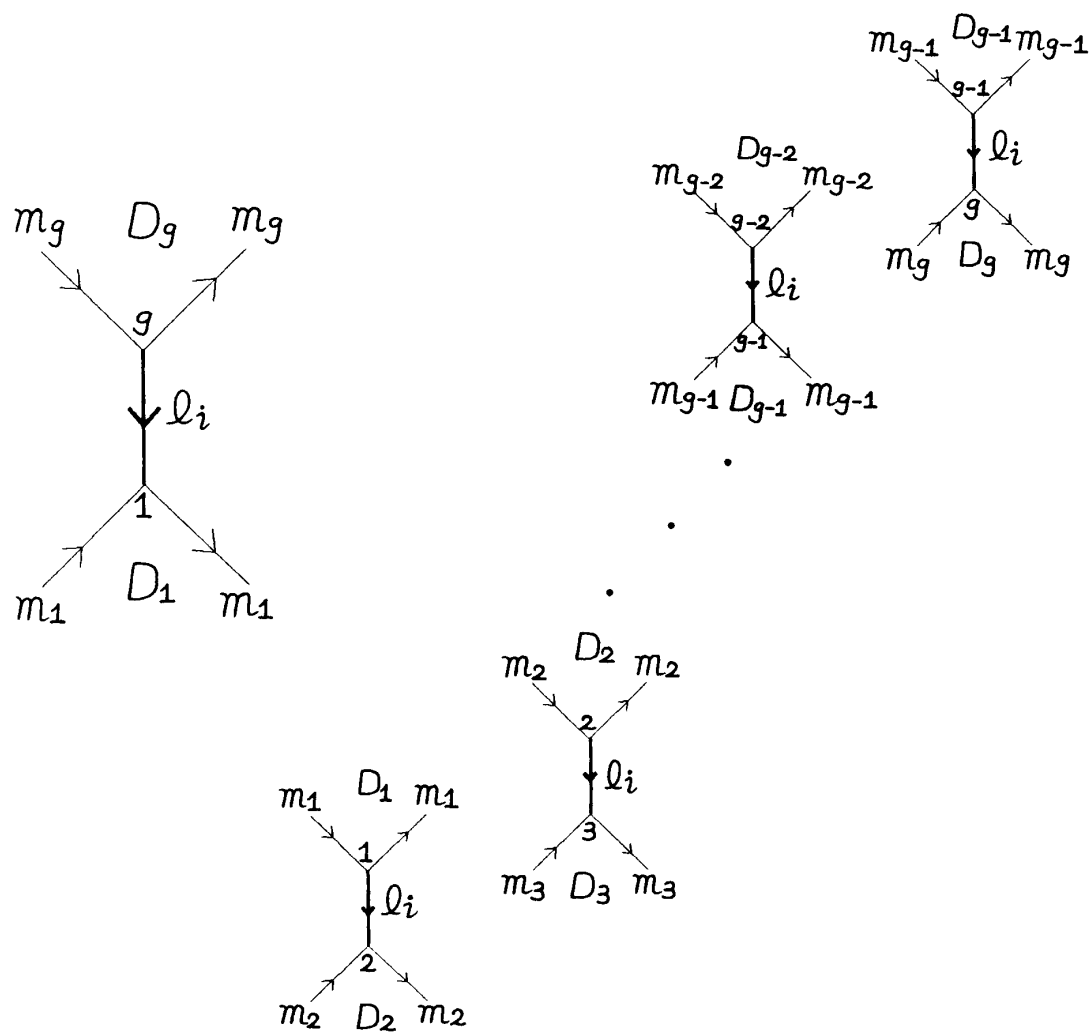
Definition 13.Table 6⁴. H_{bs} , H_e , H_d , H_l -transformation

$6-A \Rightarrow 6-B$	$H_{bs}(g+1^+, (g-2)3^+)$	$g \geq 4$
$6-B \Rightarrow 6-A$	$H_e((g-3)2^-)$	
$6-A' \Rightarrow 6-B'$	$H_d(3^-, (g-2)3^+)$	
$6-B' \Rightarrow 6-A'$	$H_l((g-3)2^-)$	

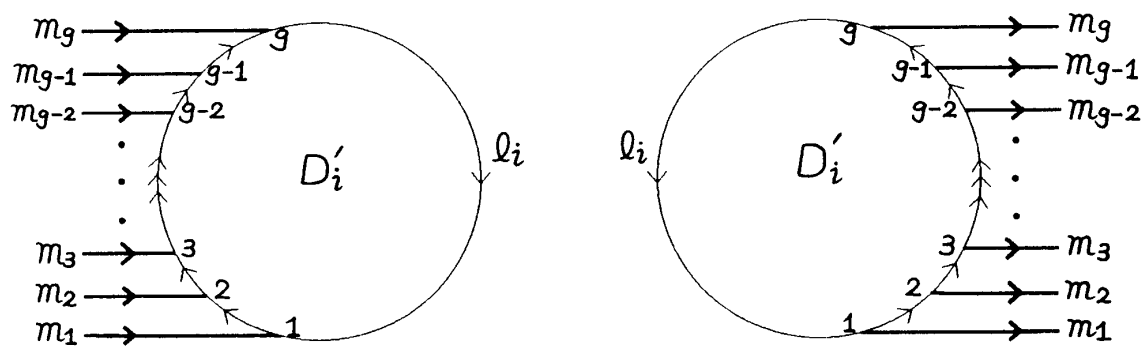
$6-A$, $6-A'$, $6-B$, $6-B'$ denotes the following figure, i.e., the part of a H-cut-diagram, without dotted lines, respectively. Later dotted lines are made use of transformations of H-cut-diagrams. $6-A'$, $6-B'$ is the dual part to $6-A$, $6-B$, respectively.

Transformation from $6-A$ ($6-A'$ resp.) to $6-B$ ($6-B'$ resp.) increases the H-genus as many as $g-3$ ($g \geq 4$), the cross point number as many as $2g-6$.

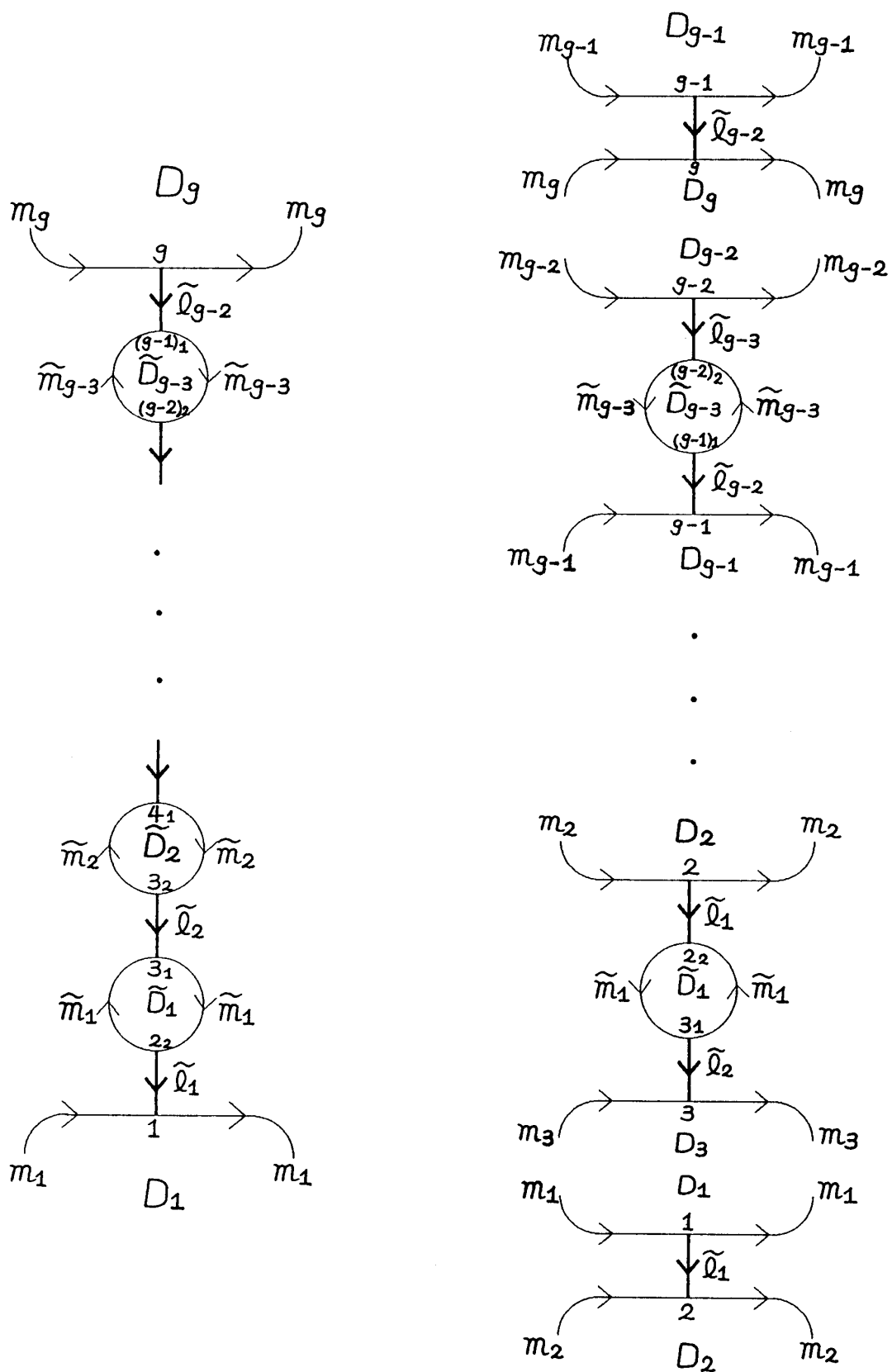
⁴ Table 6 is a serial notation from Table 5 in [10].

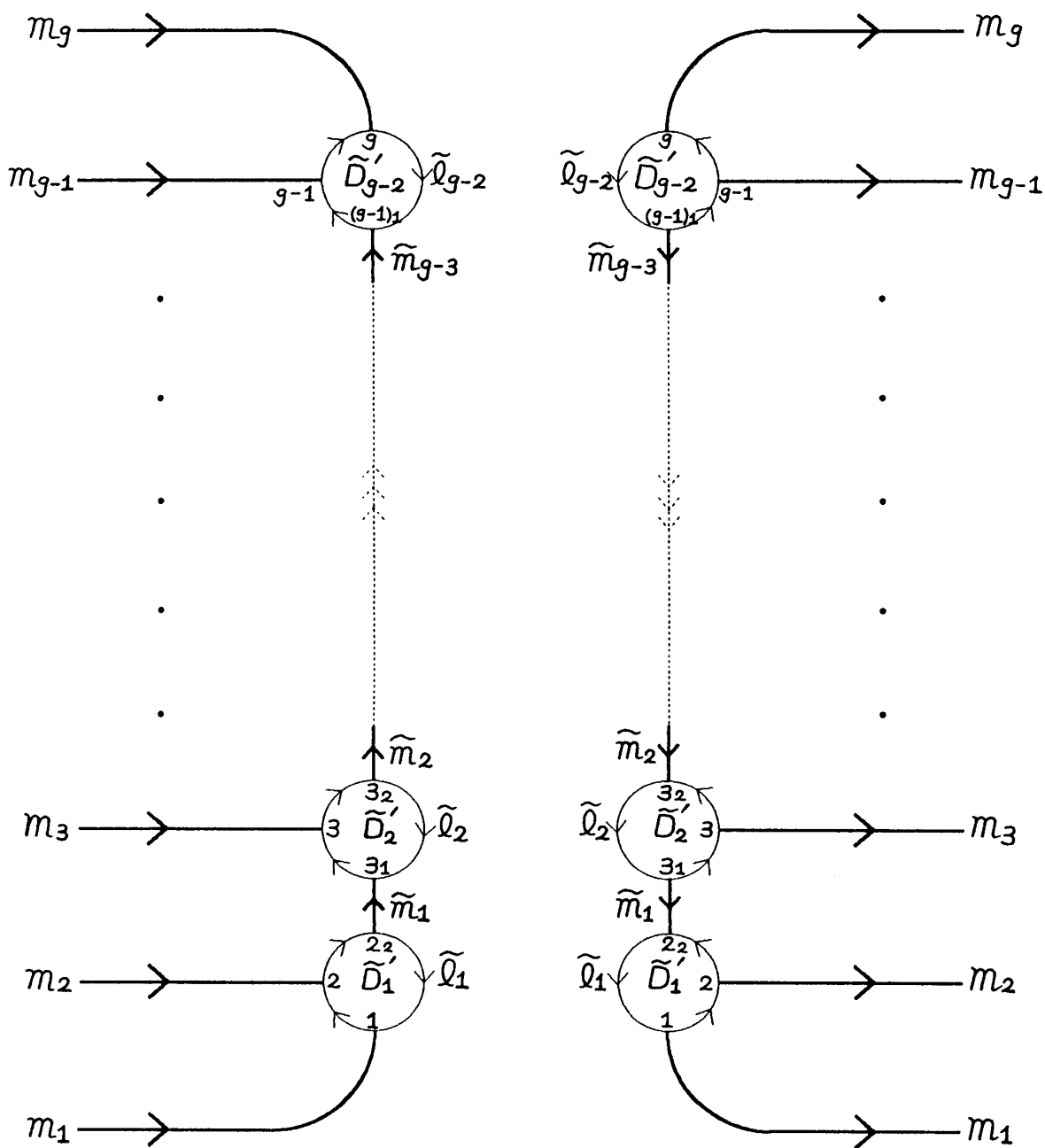


6-A



6-A'





6-B'

Let $(K ; m, l), (L ; l, m)$ be the canonical genus $g (\geq 1)$ H-diagram of (K, L, T_g) of S^3 and $G_K(m, l), G_L(l, m)$ the H-cut-diagram, respectively. Let $(U ; m, l), (V ; l, m)$ be a genus $n (\geq 1)$ H-diagram of (U, V, F) of M^3 and $G(m, l), G(l, m)$, the H-cut-diagram, respectively. If we construct the connected sum $M^3 \# S^3$ ⁵, then a corresponding H-splitting is naturally obtained. Let the notation be $(K, L, T_g) \# (U, V, F)$. Let H-diagrams of $(K, L, T_g) \# (U, V, F)$ be $(K ; m, l) \# (U ; m, l)$ and $(L ; l, m) \# (V ; l, m)$, respectively. Let the H-cut-diagrams be $G_K(m, l) \# G(m, l), G_L(l, m) \# G(l, m)$, respectively. Then $G_K(m, l) \# G(m, l) (G_L(l, m) \# G(l, m)$ resp.) becomes genus $n + g$ disconnected H-cut-diagram of $(K, L, T_g) \# (U, V, F)$ of M^3 . Then we have ;

⁵ $M^3 \# S^3$ is homeomorphic to M^3 .

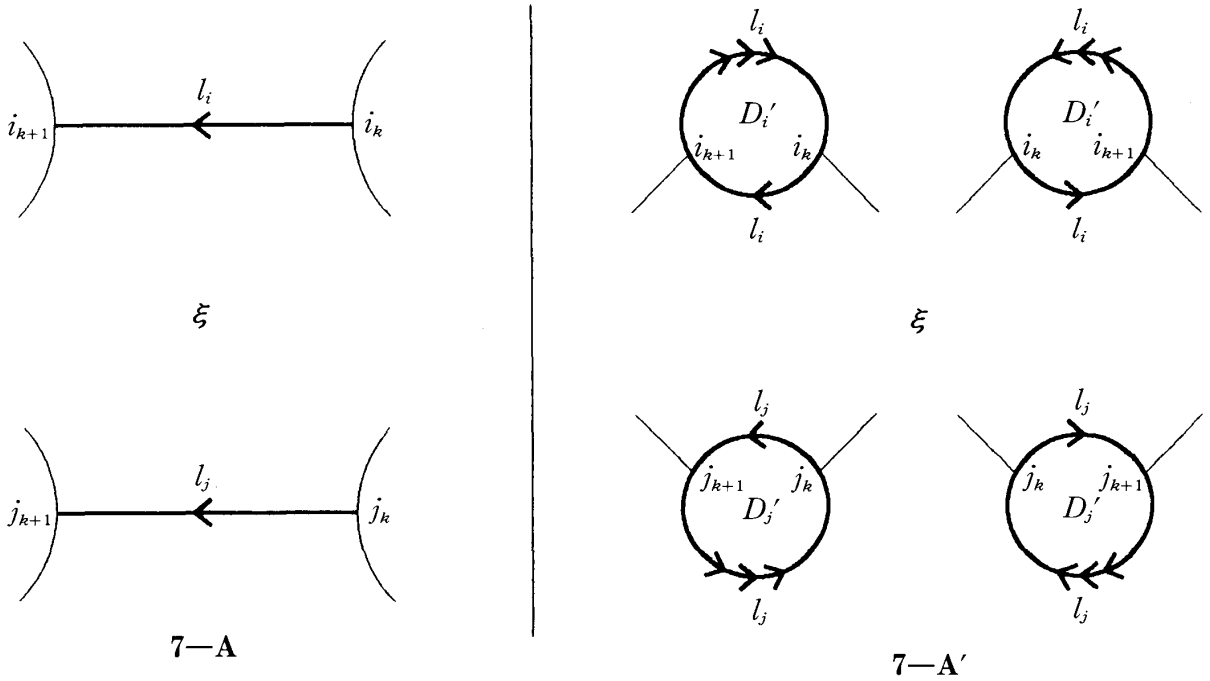
Table 7. $H^\#$ -transformation

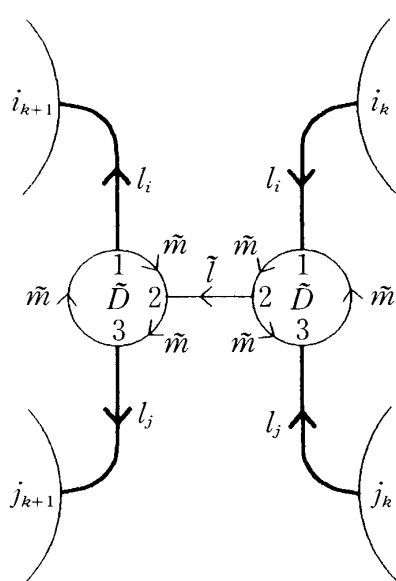
$G(m, l) \Rightarrow G_K(m, l) \# G(m, l)$	$H^\#(g^+(S^3, T_g))$
$G(l, m) \Rightarrow G_L(l, m) \# G(l, m)$	
$G_K(m, l) \# G(m, l) \Rightarrow G(m, l)$	$H^\#(g^-(S^3, T_g))$
$G_L(l, m) \# G(l, m) \Rightarrow G(l, m)$	

The transformations of $H^\#(g^+(S^3, T_g))$ increase the H-genus and the cross point number as many as g .

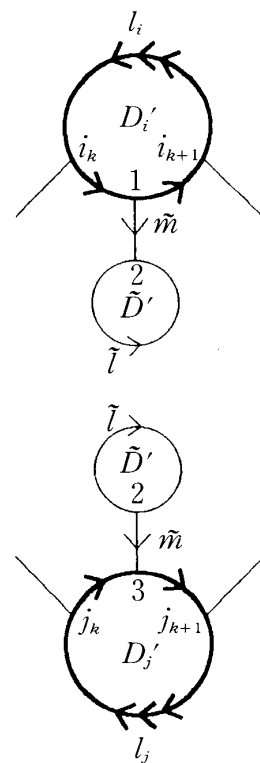
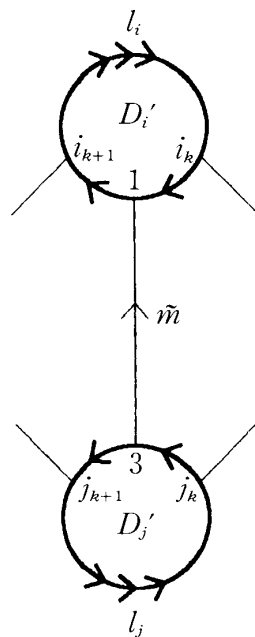
Table 8. H_t, H_b -transformation

$7-A \Rightarrow 7-B$	$H_t(2^+, 3^+)$
$7-B \Rightarrow 7-A$	$H_t(3^-, 2^-)$
$7-A' \Rightarrow 7-B'$	$H_b(2^+, 3^+)$
$7-B' \Rightarrow 7-A'$	$H_b(3^-, 2^-)$





7-B



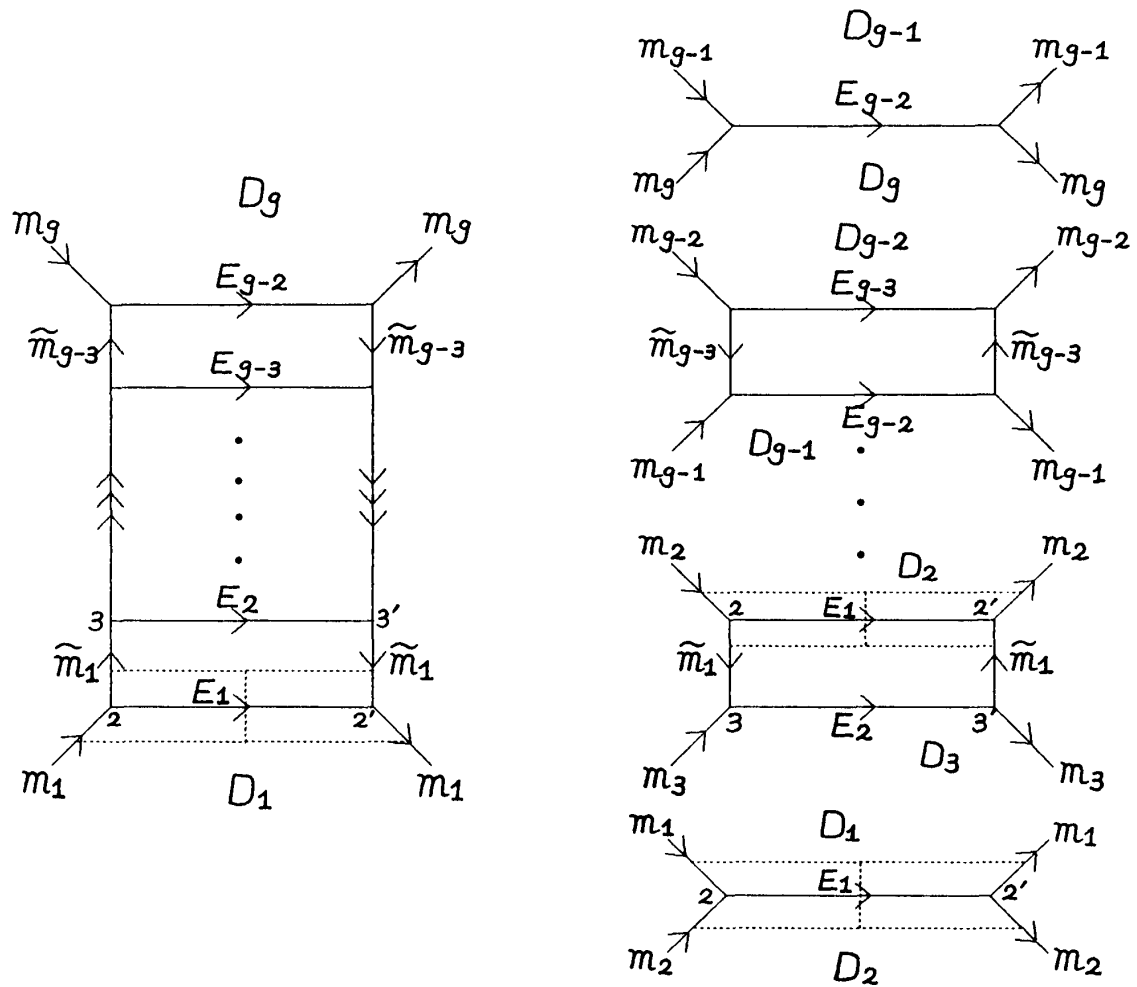
7-B'

Transformation from 7-A (7-A' resp.) into 7-B (7-B' resp.) increases the H-genus as many as 1, the cross point number as many as 3.

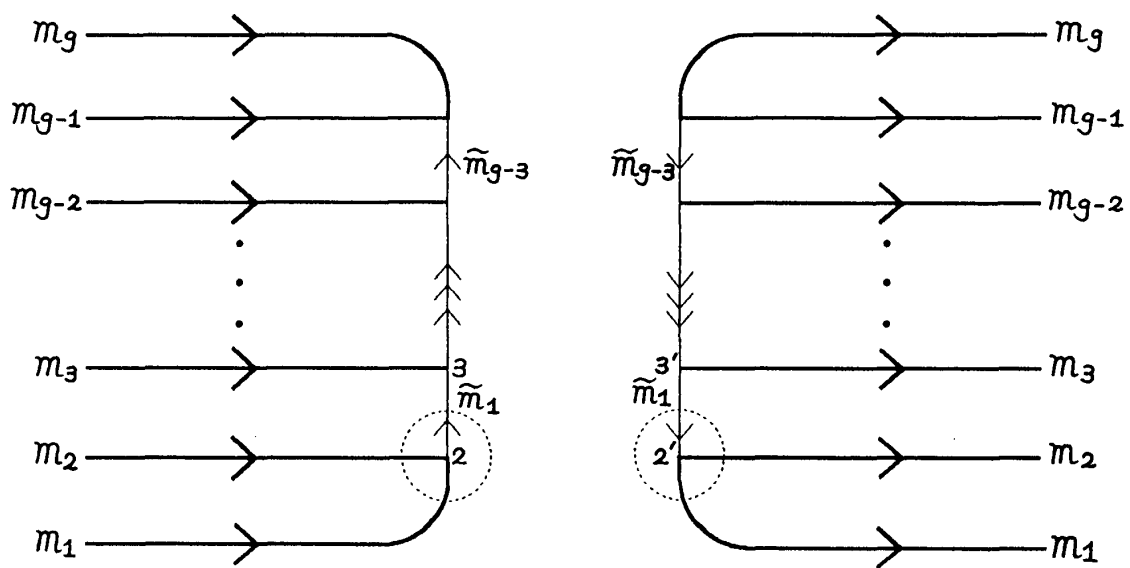
Next we illustrate the methods of the above H-cut-transformations using the DS-deformations [4].

$H_{bs}(g+1^+, (g-2)3^+)$, $H_d(3^-, (g-2)3^+)$ -transformation: $6-A \Rightarrow 6-B$ and $6-A' \Rightarrow 6-B'$

Step 1. Applying D_{g+1}^+ -deformation to $(6-A) \cup (6-A')$, the following figures $(6-A1) \cup (6-A1')$ are obtained.

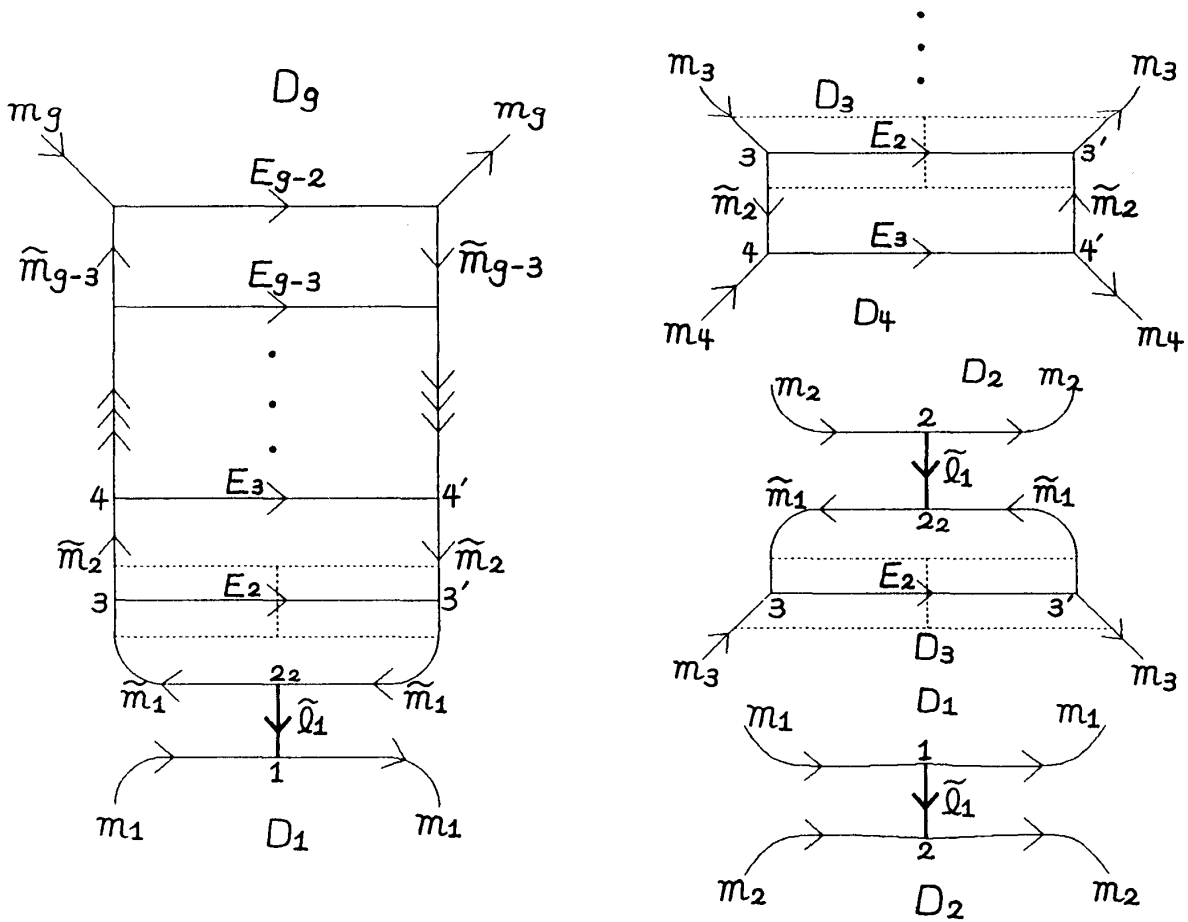


6-A1

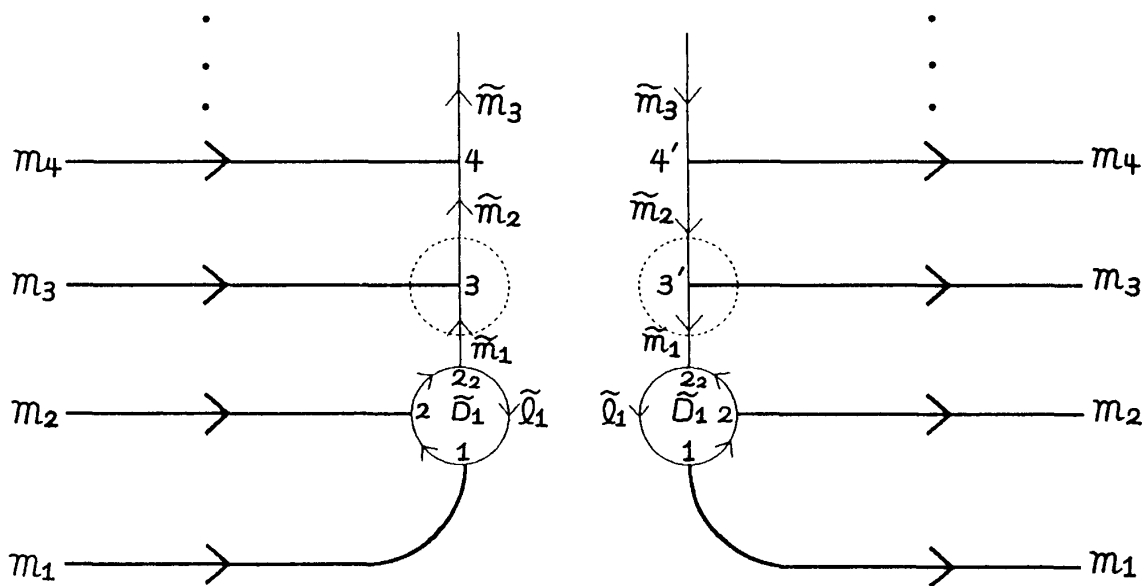


6-A1'

Step 2. Applying D_3^+ -deformation to $(6-A1) \cup (6-A1')$ shown as the six dotted lines and two circles, $(6-A2) \cup (6-A2')$ are obtained.

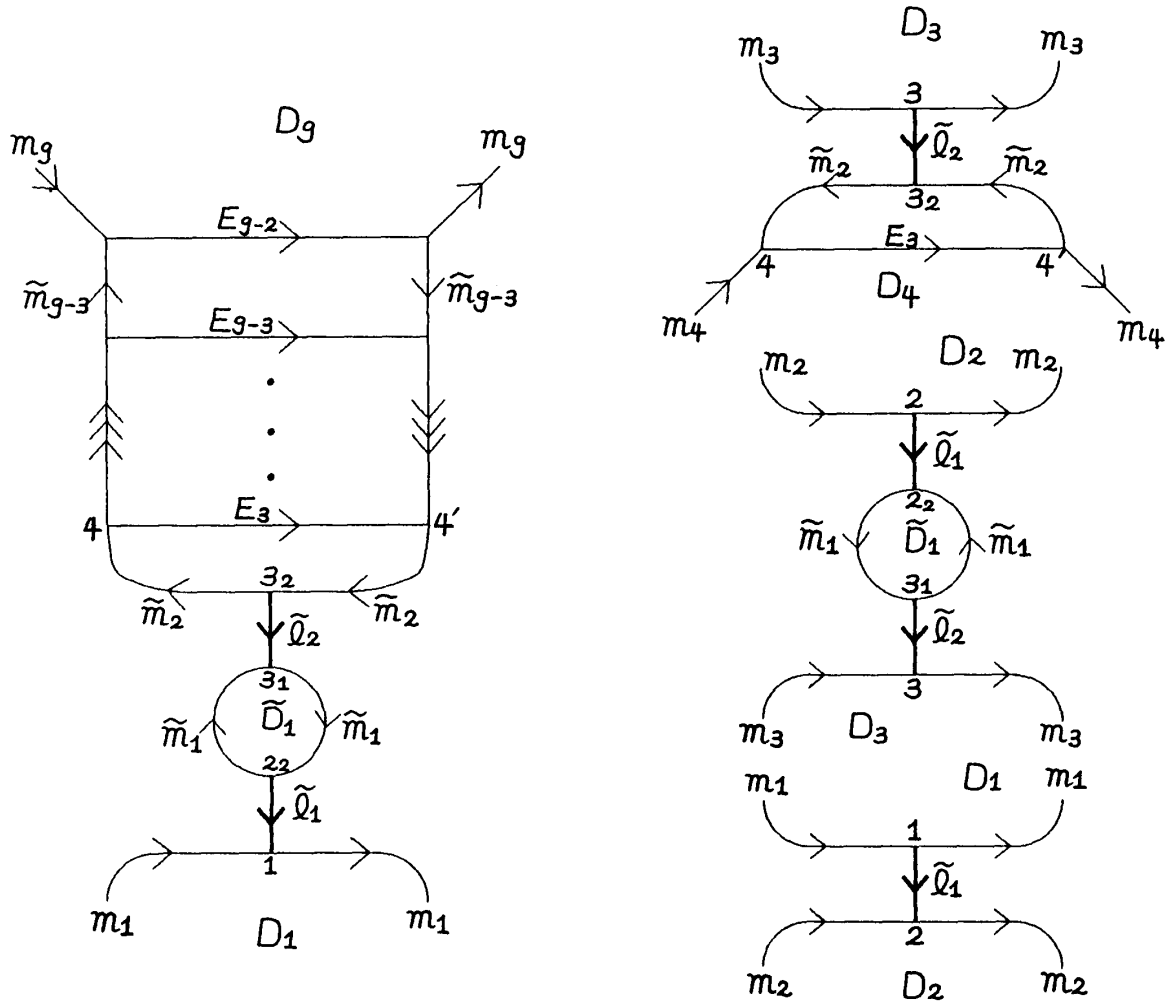


6-A2

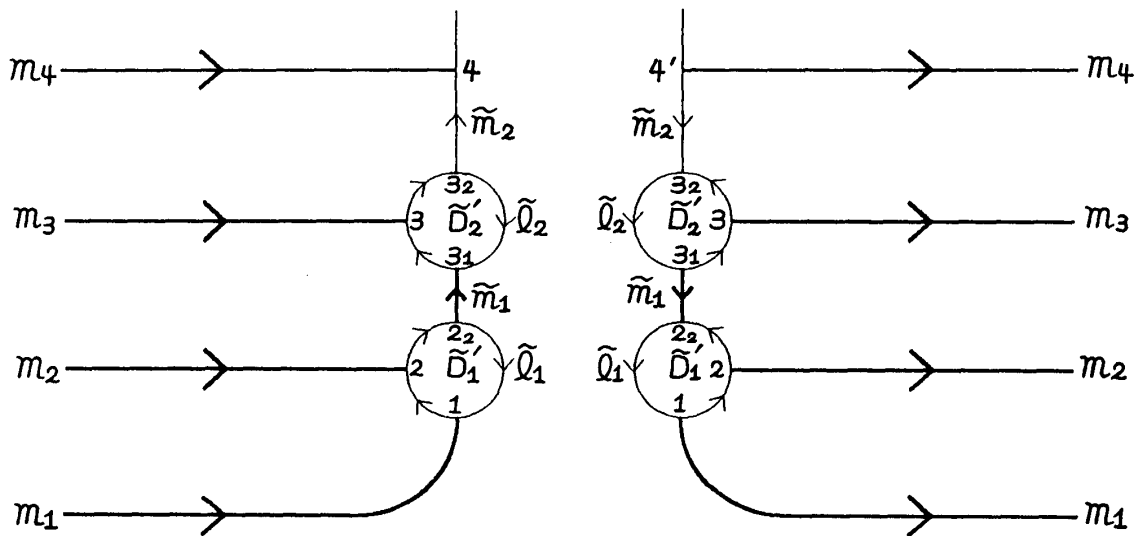


6-A2'

Step 3. Again applying D_3^+ -deformation to $(6-A2) \cup (6-A2')$ shown as the dotted lines and circles, $(6-A3) \cup (6-A3')$ are obtained.



6-A3



6-A3'

Step 4. Similarly we continue to keep D_3^+ -deformation from $(6-A3) \cup (6-A3')$ so that edge E_i ($i = 3, \dots, g-2$) is erased, then $(6-B) \cup (6-B')$ are obtained.

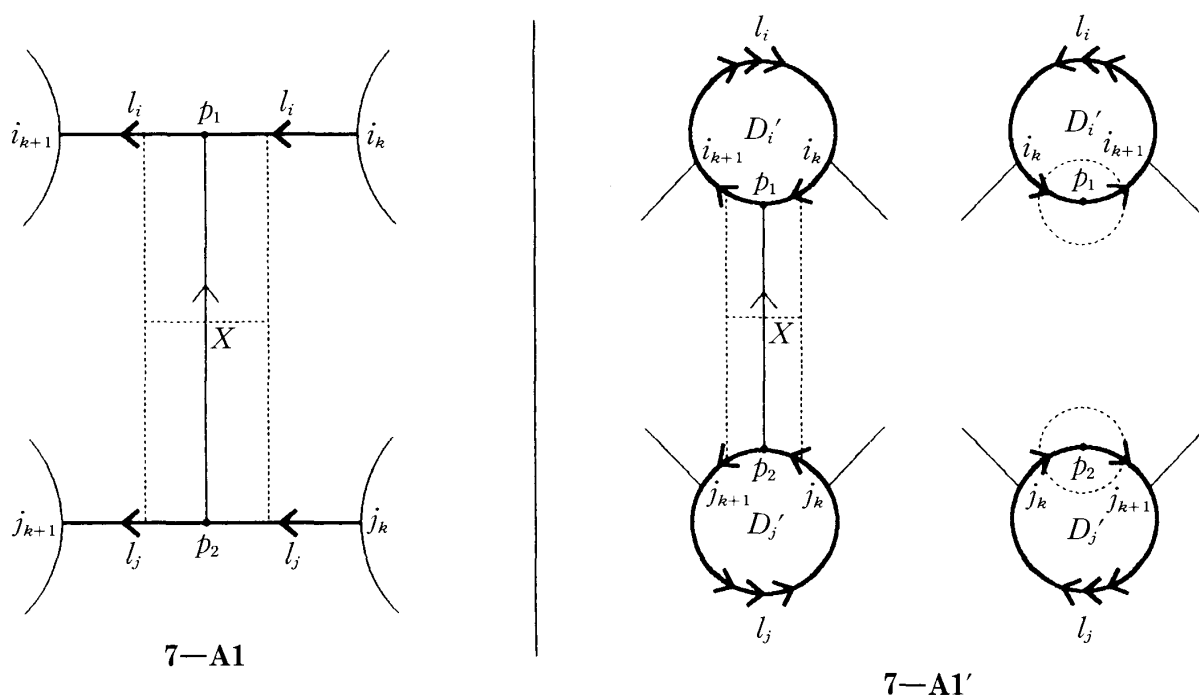
To make a transformation from $6-B$ ($6-B'$ resp.) into $6-A$ ($6-A'$ resp.), we may carry out the finite sequence of D_2^- -deformations so that those erase the disks $\{\tilde{D}_i\}$ in $6-B$.

H_i, H_b -transformation: $7-A \Rightarrow 7-B$ and $7-A' \Rightarrow 7-B'$

Step 1. Let ξ be a face (disk or punctured 2-disk⁶) in the closures of connected components of $S_{l^2} - |G(m, l)| - \cup_{j=1}^2 (D_j \cup D_j')$. ξ satisfies the following condition.

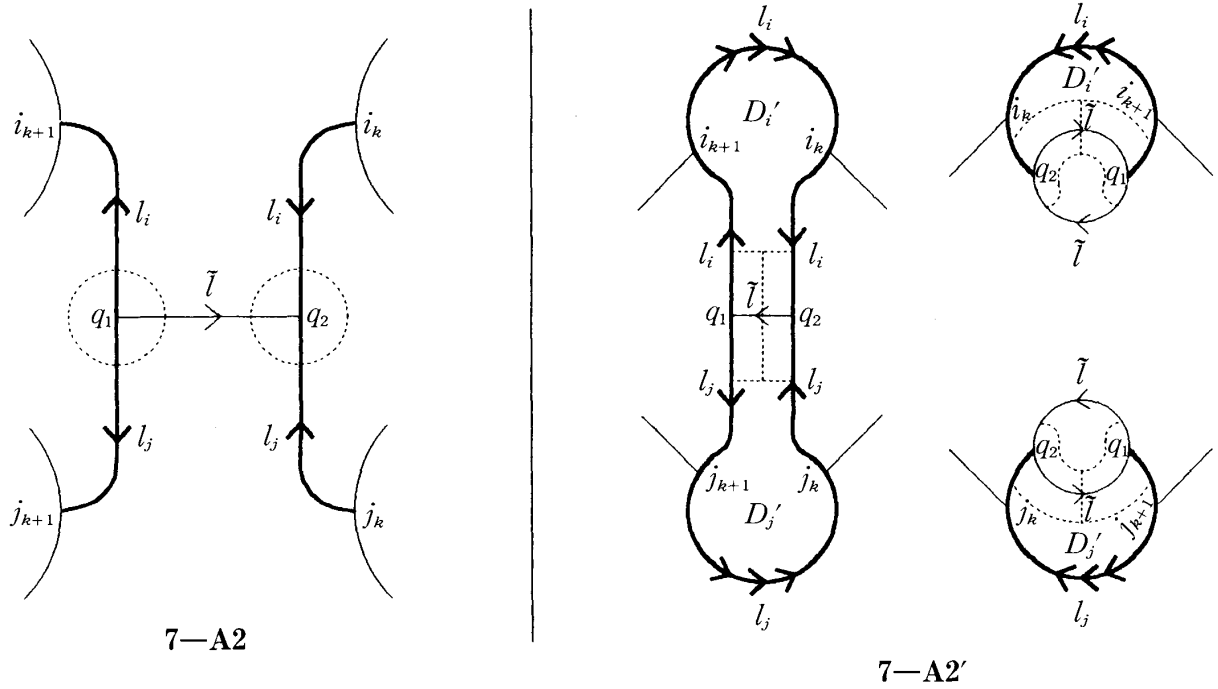
(C) $i_k(l_i)i_{k+1} \subset \partial\xi$ and $j_k(l_j)j_{k+1} \subset \partial\xi$ ($i \neq j$)

Step 2. There exists the same labeled face ξ in $G(l, m)$ that is inversely oriented to that in $G(m, l)$. Draw the same oriented labeled edges X in both ξ so that $p_1 \subset \text{Int}(i_k(l_i)i_{k+1}) \subset \partial\xi$ and $p_2 \subset \text{Int}(j_k(l_j)j_{k+1}) \subset \partial\xi$ where $\{p_1, p_2\}$ are two points of ∂X . $7-A1$ ($7-A1'$ resp.) denotes $\{G(m, l) \cup X\}$ ($\{G(l, m) \cup X\}$ resp.).



⁶ disk with n (≥ 1) holes.

Step 3. Carry out D_2^+ -deformation shown as the four dotted lines and two circles in $\{G(m, l) \cup X\} \cup \{G(l, m) \cup X\}$, then deformed diagrams 7-A2 and 7-A2' are obtained.



Step 4. Carry out D_3^+ -deformation to $q_1(\tilde{l})q_2$ shown as the dotted lines and circles in (7-A2) \cup (7-A2'), then (7-B) \cup (7-B') are obtained.

Remark 1. In (C), we may change $\{i_k(m_i)i_{k+1}, j_k(m_j)j_{k+1}\} (i \neq j)$ instead of $\{i_k(l_i)i_{k+1}, j_k(l_j)j_{k+1}\}$ but if we take a different pair $\{i_k(l_i)i_{k+1}, i_k(m_i)i_{k+1}\}$ of a longitude and meridian instead of $\{i_k(l_i)i_{k+1}, j_k(l_j)j_{k+1}\}$, then H-cut-diagrams can not be constructed from $\{G(m, l) \cup X\} \cup \{G(l, m) \cup X\}$.

Remark 2. The diagrams shown in (7-A1) \cup (7-A1')((7-A2) \cup (7-A2')) are not H-cut-diagrams but they represent M^3 .

We consider in the case that ξ does not satisfy the condition (C). Let such ξ be η . Let $(K; m, l), (L; l, m)$ be the canonical genus 1 H-diagram of (K, L, T_1) of S^3 and $G_K(m, l), G_L(l, m)$ the H-cut-diagram, respectively. We draw $G_K(m, l) (G_L(l, m) \text{ resp.})$ in $\text{Int}(\eta) (\subset |G(m, l)|) (\text{Int}(\eta) (\subset |G(l, m)|) \text{ resp.})$. Then $G_K(m, l) \cup G(m, l) (G_L(l, m) \cup G(l, m) \text{ resp.})$ becomes a H-cut-diagram $G_K(m, l) \# G(m, l) (G_L(l, m) \# G(l, m) \text{ resp.})$ of $(K; m, l) \# (U; m, l) ((L; l, m) \# (V; l, m) \text{ resp.})$ of connected sum $M^3 \# S^3$. From $G_K(m, l) \cup \eta$ and $G_L(l, m) \cup \eta$ we may find the faces such as ξ in (7-A) \cup (7-A'). Therefore carrying out the same transformations as in the above (step 1~4), the parts of (7-B) \cup (7-B') that increase the H-genus as many as 2 are obtained from $(G_K(m, l) \# G(m, l)) \cup (G_L(l, m) \# G(l, m))$.

Using the above transformations of (7-A) \cup (7-A') \Rightarrow (7-B) \cup (7-B') repeatedly, we have ;

Proposition 3. *Let $G(m, l)$ ($G(l, m)$ resp.) be a disconnected genus n (≥ 1) H-cut-diagram of (U, V, F) of M^3 . Then there exists a transformation from the disconnected H-cut-diagram into connected one.*

Proof. We give an algorithm of construction as follows.

Step 1. Since $G(m, l)$ is disconnected, there exists a punctured 2-disk in $G(m, l)$. Let ξ be punctured 2-disks in both $G(m, l)$ and $G(l, m)$. If ξ does not satisfy the condition (C), then we construct the connected sum $M^3 \# S^3$ such as given in front of this Prop.

Step 2. We draw the same oriented labeled edges X_1, \dots, X_t in both ξ that satisfy the following conditions :

- (1) Let $\{p_{i_1}, p_{i_2}\}$ be two points of ∂X_i and $\{p_{i_1}, p_{i_2}\}$ ($i = 1, \dots, t$) differently.
- (2) $p_{i_1} \subset \text{Int}(j_{i_1}(l_{j_{i_1}})j_{i_1+1}) \subset \partial \xi$ and $p_{i_2} \subset \text{Int}(j_{i_2}(l_{j_{i_2}})j_{i_2+1}) \subset \partial \xi$.
- (3) The closure of the connected component of $\xi - (X_1 \cup \dots \cup X_t)$ becomes 2-disk.

Step 3. Carry out the transformations from (7-A) \cup (7-B) into (7-A') \cup (7-B') to both parts of the X_1 in $\{G(m, l) \cup X_1\} \cup \{G(l, m) \cup X_1\}$. Then we get genus $n+1$ or $n+2$ H-cut-diagrams.

Step 4. Carry out the same operations as step 3 to other X_i ($i = 2, \dots, t$), then we obtain H-cut-diagrams in which genres are more than $n+t-1$. Let the H-cut-diagrams be $G'(m, l) \cup G'(l, m)$.

Step 5. If $G'(m, l)$ is not connected, carry out the same operations from step 1 to 4 to $G'(m, l) \cup G'(l, m)$, repeatedly, then connected H-cut-diagrams are obtained. \square

Let $(U_i; m, l), (V_i; l, m)$ ($i = 1, 2$) be genus n_i (≥ 1) H-diagram of (U_i, V_i, F_i) of M_i^3 and $G_{U_i}(m, l), G_{V_i}(l, m)$ the H-cut-diagram, respectively. If we construct the connected sum $M_1^3 \# M_2^3$, then we get disconnected genus $n_1 + n_2$ H-diagrams $(U_1; m, l) \# (U_2; m, l), (V_1; l, m) \# (V_2; l, m)$ of $(U_1, V_1, F_1) \# (U_2, V_2, F_2)$ and their disconnected H-cut-diagrams $G_{U_1}(m, l) \# G_{U_2}(m, l), G_{V_1}(l, m) \# G_{V_2}(l, m)$. Then we have ;

Proposition 4. *There exists a transformation from the disconnected H-cut-diagram $G_{U_1}(m, l) \# G_{U_2}(m, l)$ ($G_{V_1}(l, m) \# G_{V_2}(l, m)$ resp.) of $(U_1, V_1, F_1) \# (U_2, V_2, F_2)$ of $M_1^3 \# M_2^3$ into connected one.*

4. Proofs of Theorem 3

Proofs of (7) and (7'). The H-cut-diagrams derived from U6-A (V6-A' resp.) and U6-B (V6-B' resp.) are 6-A (6-A' resp.) and 6-B (6-B' resp.), respectively. Therefore the transformation from 6-A (6-A' resp.) into 6-B (6-B' resp.) gives a transformation from U6-A (U6-A' resp.) into U6-B (U6-B'). \square

Proofs of (8) and (8'). The transformation from 7-A (7-A' resp.) into 7-B (7-B' resp.) gives a transformation from U7-A (U7-A' resp.) into U6-B (U6-B'). \square

Definition 14. The transformations of theorem 2 ([10]) and 3 will be generally called the *H-transformations for Heegaard diagrams*.

Relations between the operations used for the transformations of H-diagrams and DS-deformations used for those of H-cut-diagrams are a theorem stated as the following table.

Theorem 4.

Transformation of H-diagram	Operation for H-diagram	Transformation of H-cut-diagram	DS-deformation
$U6-A \Rightarrow U6-B$	or $g-3$ handles adding handlebody cutting by $g-3$ annuluses	$6-A \Rightarrow 6-B$	$D_{g+1}^+ \rightarrow$ $g-2$ times of D_3^+
$V6-A' \Rightarrow V6-B'$	and handle cutting $g-2$ handles adding	$6-A' \Rightarrow 6-B'$	
$U7-A \Rightarrow U7-B$	or handle adding handlebody cutting by an annulus	$7-A \Rightarrow 7-B$	$D_3^+ \rightarrow D_3^+$
$V7-A' \Rightarrow V7-B'$	or handle adding handlebody cutting by an annulus	$7-A' \Rightarrow 7-B'$	

Further we derive the following H-transformations from Prop. 3, 4, respectively.

Proposition 5. *Let $(U; m, l)$ ($(V; l, m)$ resp.) be a disconnected genus n (≥ 1) H-diagram of (U, V, F) of M^3 . Then there exists a transformation from the disconnected H-diagram into connected one.*

Proposition 6. *There exists a transformation from a disconnected H-diagram $(U_1; m, l) \# (U_2; m, l)$ ($(V_1; l, m) \# (V_2; l, m)$ resp.) of $(U_1, V_1, F_1) \# (U_2, V_2, F_2)$ of $M_1^3 \# M_2^3$ into connected one.*

5. Some examples

Example 4. We begin with the definition of a wave.

Definition 15. Let σ_i be a disk in the closures of connected components of $S_l^2 - |G(m, l)| - \cup_{j=1}^n (D_j \cup D_j)$. $\partial\sigma_i$ consists of a part of edges in $G(m, l)$. Suppose that there exist oriented two edges $j_\alpha(m_j)j_{\alpha+1}$, $j_\beta(m_j)j_{\beta+1}$ ($j_\alpha(l_j)j_{\alpha+1}$, $j_\beta(l_j)j_{\beta+1}$ resp.) ($\alpha \neq \beta$) of $m_j(l_j)$ resp.) in $\partial\sigma_i$. When we go round the circle $\partial\sigma_i$ clockwise or counterclockwise, suppose that the orientations of both $j_\alpha(m_j)j_{\alpha+1}$ and $j_\beta(m_j)j_{\beta+1}$ ($j_\alpha(l_j)j_{\alpha+1}$ and $j_\beta(l_j)j_{\beta+1}$ resp.) become the same as that. Let w be an arc in σ_i and ∂w points p_1, p_2 such as $p_1 \subset \text{Int}(j_\alpha(m_j)j_{\alpha+1})$ ($\text{Int}(j_\alpha(l_j)j_{\alpha+1})$ resp.) and $p_2 \subset \text{Int}(j_\beta(m_j)j_{\beta+1})$ ($\text{Int}(j_\beta(l_j)j_{\beta+1})$ resp.). Then w is called a wave of $G(m, l)$. Similarly a wave of $G(l, m)$ is defined.

By the definition of a wave, if $G(m, l)$ has waves $\{w_1, w_2, \dots, w_\alpha\}$, then $G(l, m)$ has the same waves as $G(m, l)$.

The Whitehead [11]-Volodin-Kuznetsov-Fomenko [12] conjecture shows that “all H-diagrams of the 3-sphere S^3 other than the canonical one have waves without fail.” This is an algorithm for recognizing S^3 in 3-manifold. In [14], Birman says that “nobody has succeeded in verifying such an assertion between 1935 and 1977, or producing a counter example.” In 1980, Homma-Ochiai-Takahashi [17] success in the above conjecture if H-genus = 2 but Viro [13], Morikawa [18] and Ochiai [19] construct counter examples if H-genus ≥ 3 .

A counter example by a transformation from waves H-cut-diagram into no-waves one is obtained in [9, p. 76]. Fig. 2 gives genus 2 H-diagram $(U; m, l, 7)$, $(V; l, m, 7)$ of S^3 and those H-cut-diagrams $G_1(m, l, 7)$, $G_1(l, m, 7)$, respectively. Here $U = B_U^3 + (h_1 \cup h_2)$ and $V = B_V^3 + (h_1' \cup h_2')$. The number 7 of $(U; m, l, 7)$ ($G_1(m, l, 7)$ resp.) indicates the cross point number of $m \cap l$. $G_1(m, l, 7)$, $G_1(l, m, 7)$ has two waves drawn by wavy lines, respectively. Here $Cl(\partial B_U^3 - \cup_{i=1}^2 \{D_i \times 0, D_i \times 1\})$ ($Cl(\partial B_V^3 - \cup_{i=1}^2 \{D_i' \times 0, D_i' \times 1\})$ resp.) is the H-cut-diagram $G_1(m, l, 7)$ ($G_1(l, m, 7)$ resp.).

From now on we carry out transformations of the above H-diagrams $(U; m, l, 7) \cup (V; l, m, 7)$ corresponding to those from waves H-cut-diagrams $G_1(m, l, 7) \cup G_1(l, m, 7)$ into no-waves ones given in [9].

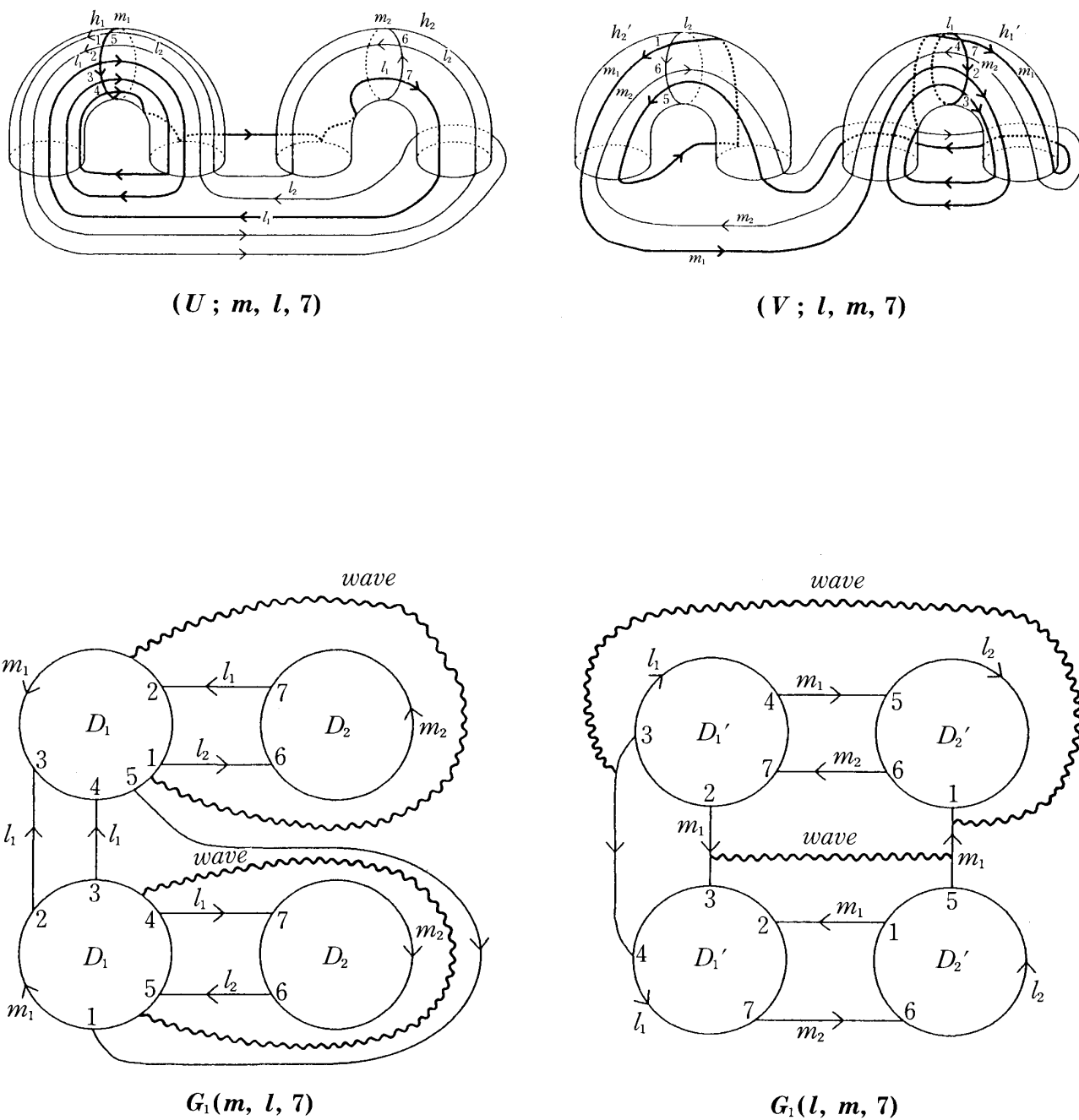
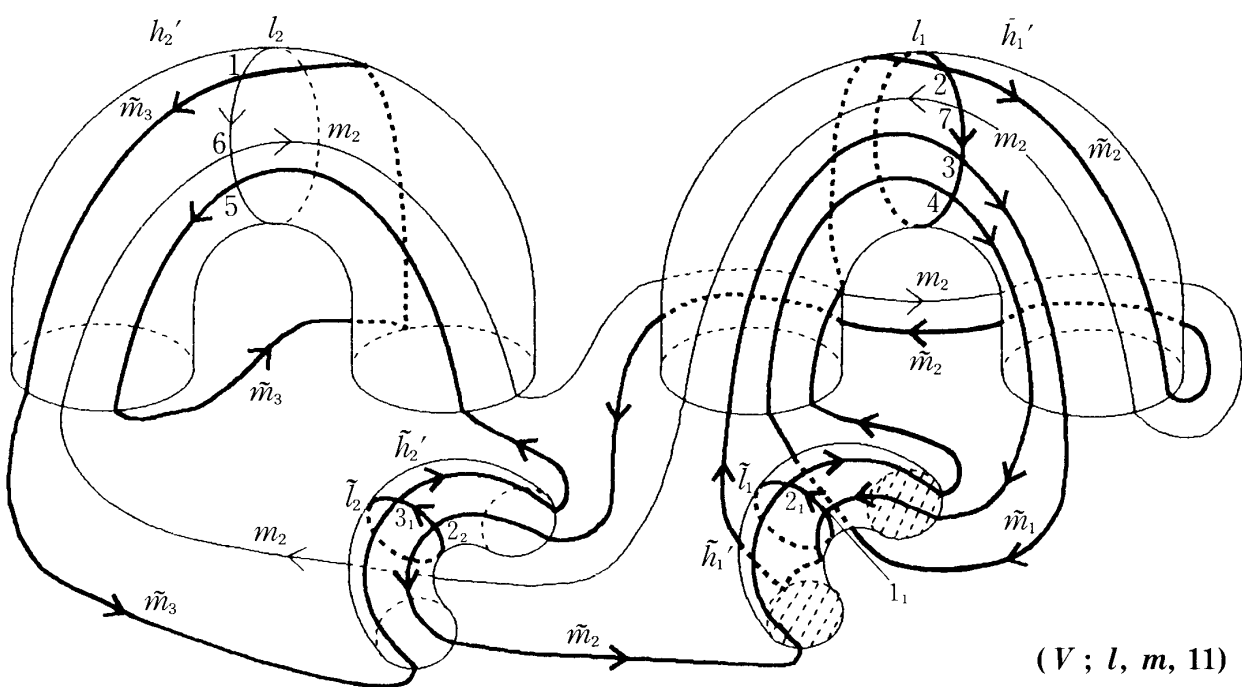
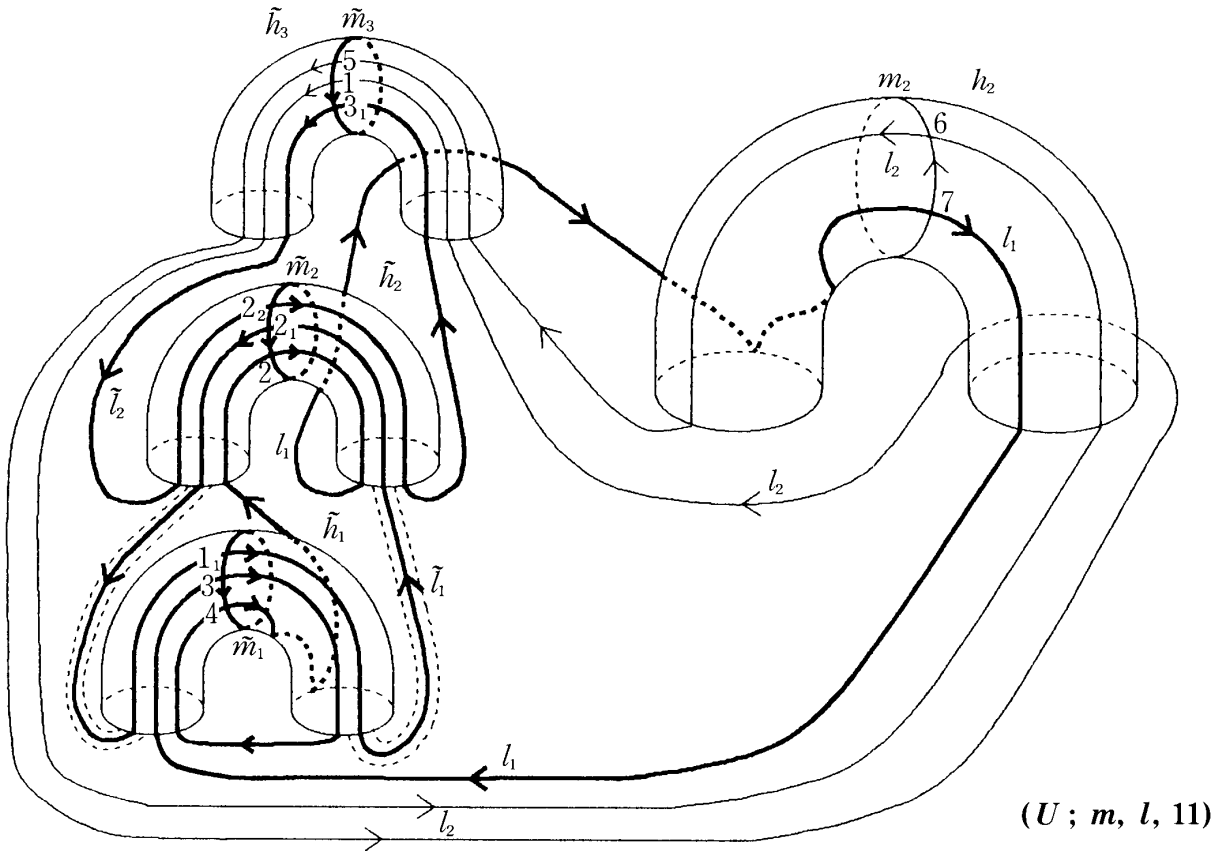


Fig. 2

Tfm $(U ; m, l, 7)$ (Transformation of $(U ; m, l, 7)$); cutting off the handle h_1 at the meridian disk $D_i (\partial D_i = m_i)$ and putting three handles \tilde{h}_1 , \tilde{h}_2 and \tilde{h}_3 over the two cutting places, $(U ; m, l, 11)$ of genus 4 is obtained.

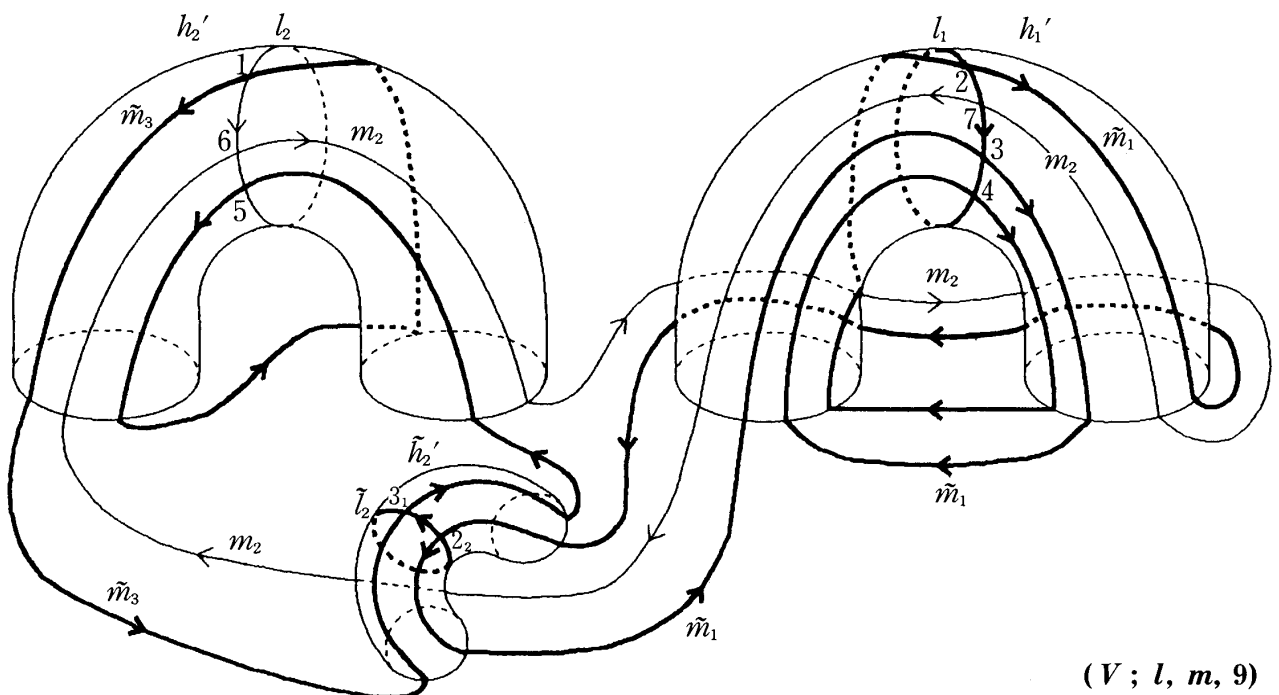
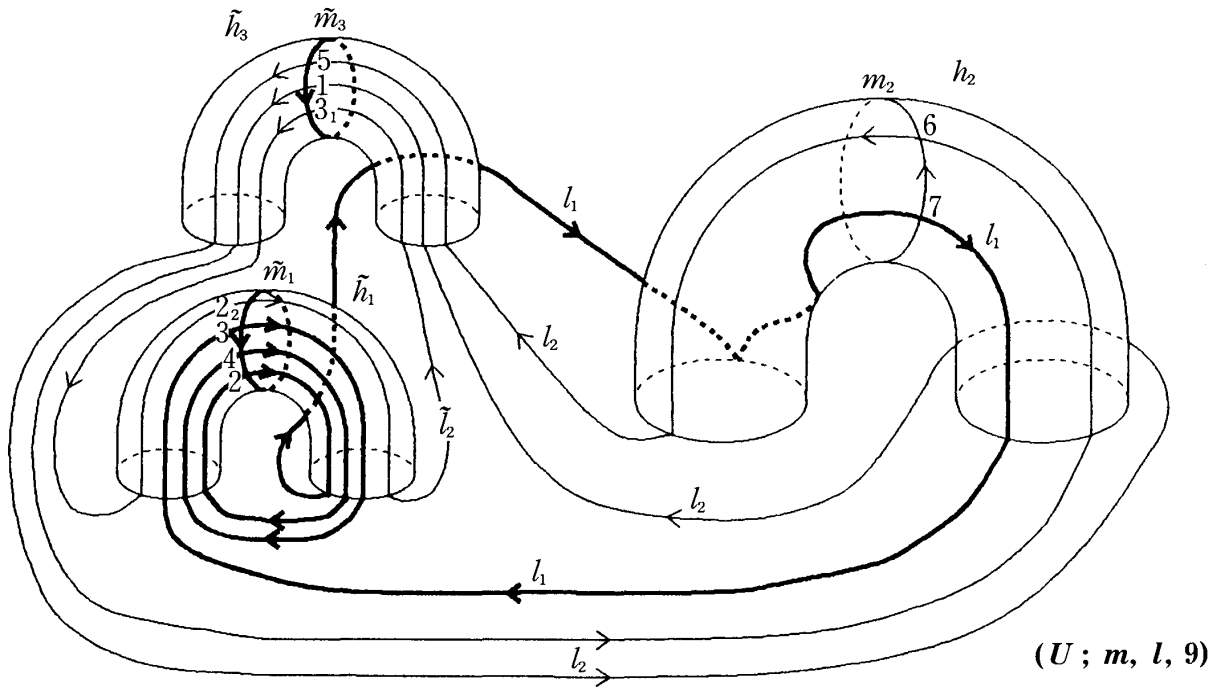
Tfm $(V ; l, m, 7)$; adding two handles \tilde{h}_1' and \tilde{h}_2' to V , $(V ; l, m, 11)$ of genus 4 is obtained.



The H-cut-diagrams $G_2(m, l, 11) \cup G_2(l, m, 11)$ which are omitted here of $(U; m, l, 11) \cup (V; l, m, 11)$ have no-waves, respectively.

Tfm $(V; l, m, 11)$; crushing the handle \tilde{h}_1' at the meridian disk \tilde{D}_1' ($\partial\tilde{D}_1' = \tilde{l}_1$) and cutting the deformed handle at the crushed disk (a pint), a genus 3 handlebody and a new circle is obtained. For the details of this way, refer to [10, p. 43]. Next putting the label \tilde{m}_1 on the circle and reorienting \tilde{m}_1 , $(V; l, m, 9)$ of genus 3 is obtained.

Tfm $(U; m, l, 11)$; genus 3 H-diagram $(U; m, l, 9)$ is obtained corresponding to the above transformation.



Dotted lines in $(U; m, l, 11) \cup (V; l, m, 11)$ show the method of transformations of the H-cut-diagrams corresponding to those of $(U; m, l, 11) \cup (V; l, m, 11)$.

The H-cut-diagrams $G_3(m, l, 9) \cup G_3(l, m, 9)$ of $(U; m, l, 9) \cup (V; l, m, 9)$ are Viro's case.

Example 5. Fig. 3 gives disconnected genus 1 H-diagrams $(U; m, l, 0)$, $(V; l, m, 0)$ of $S^2 \times S^1$ and those disconnected H-cut-diagrams $G_1(m, l, 0)$, $G_1(l, m, 0)$, respectively.

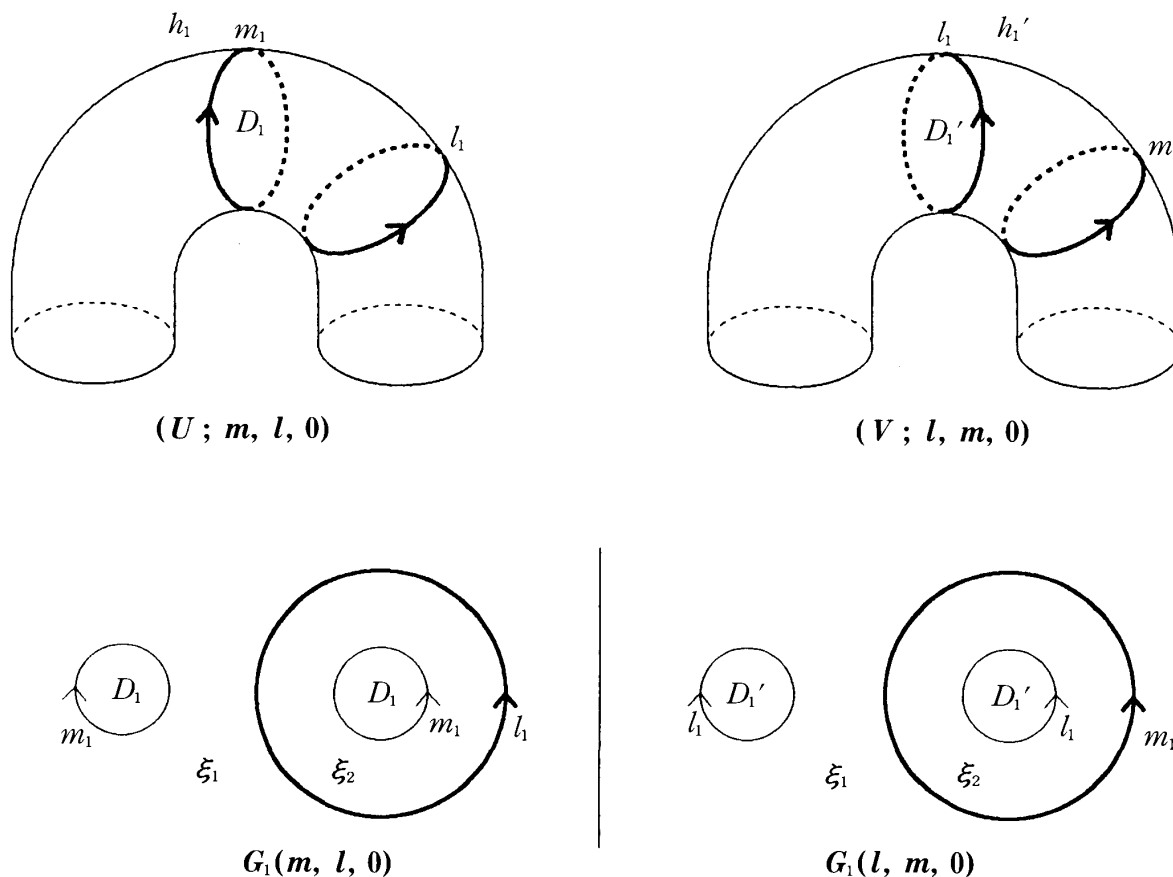
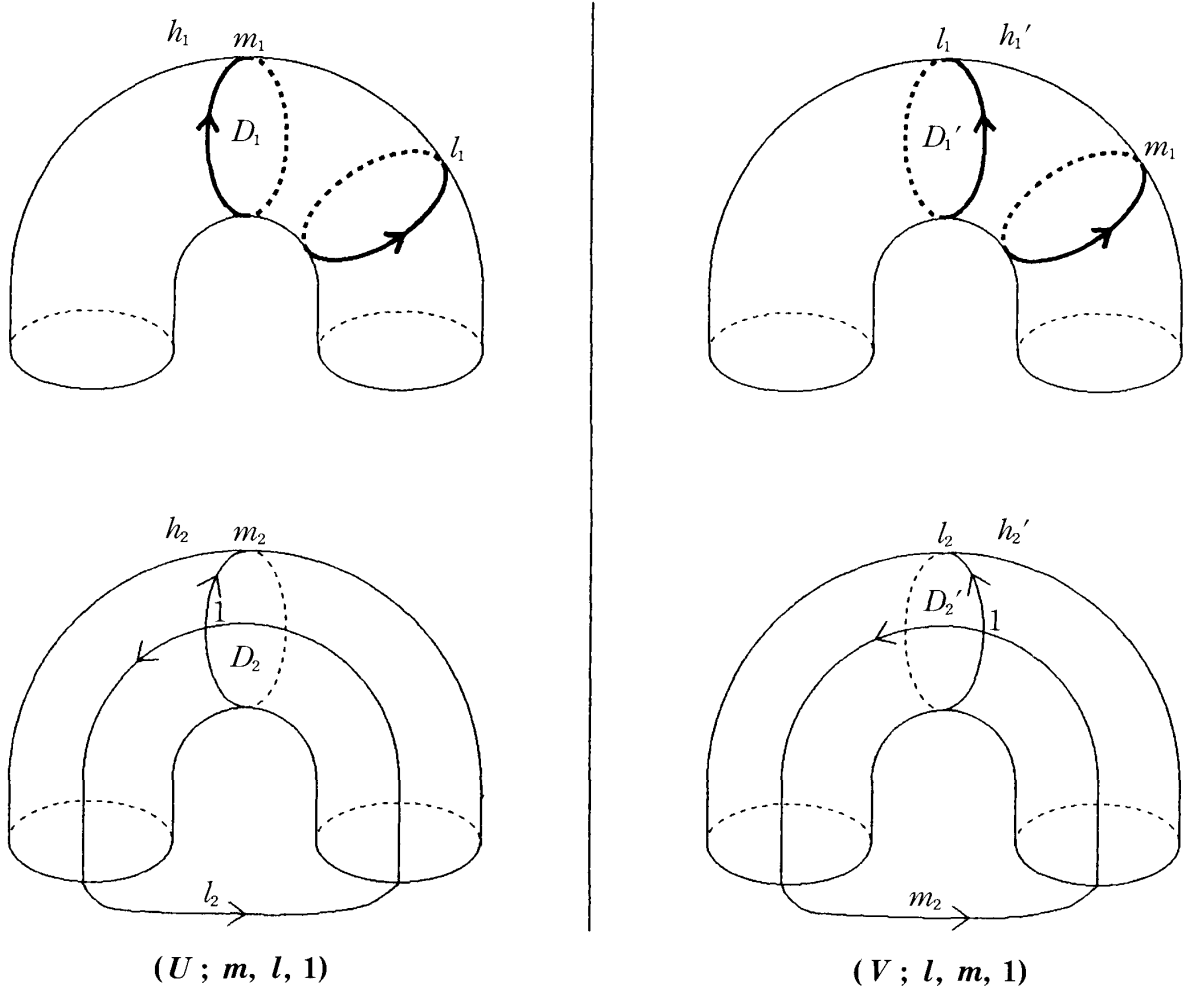


Fig. 3

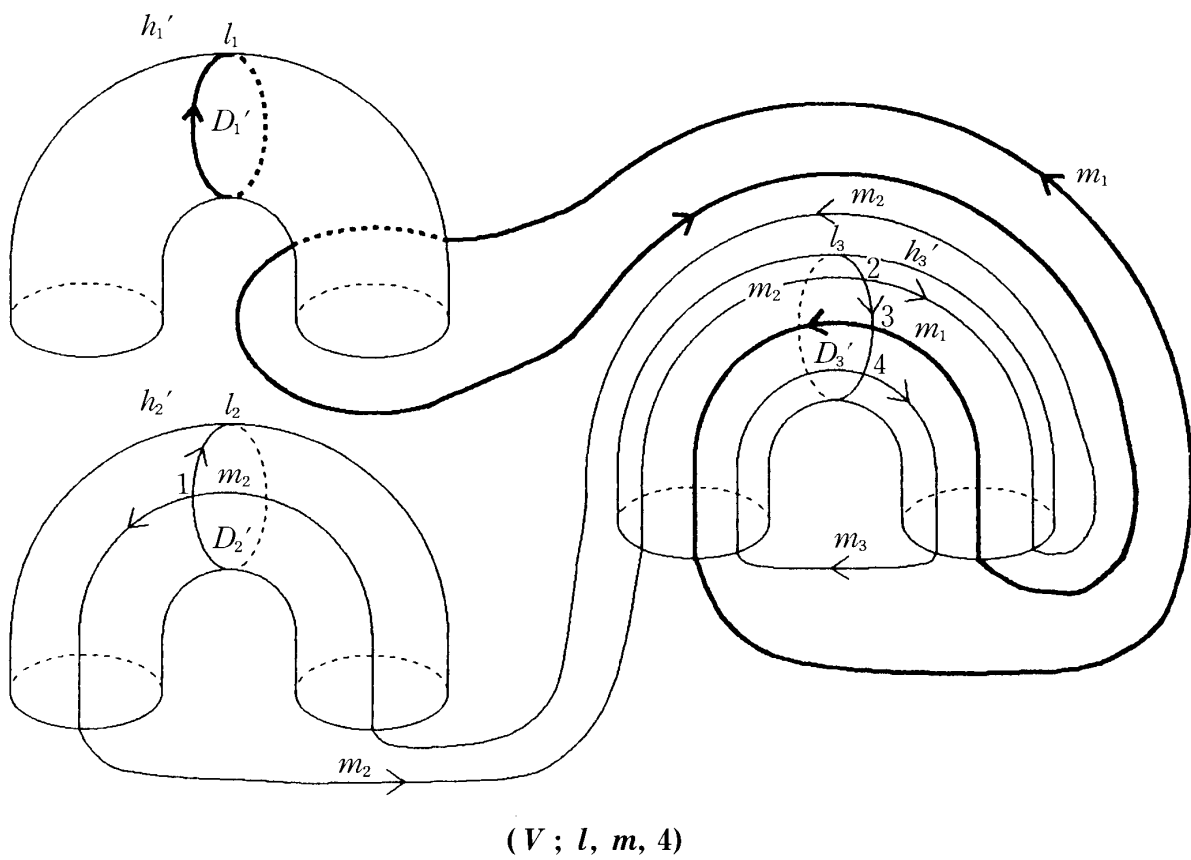
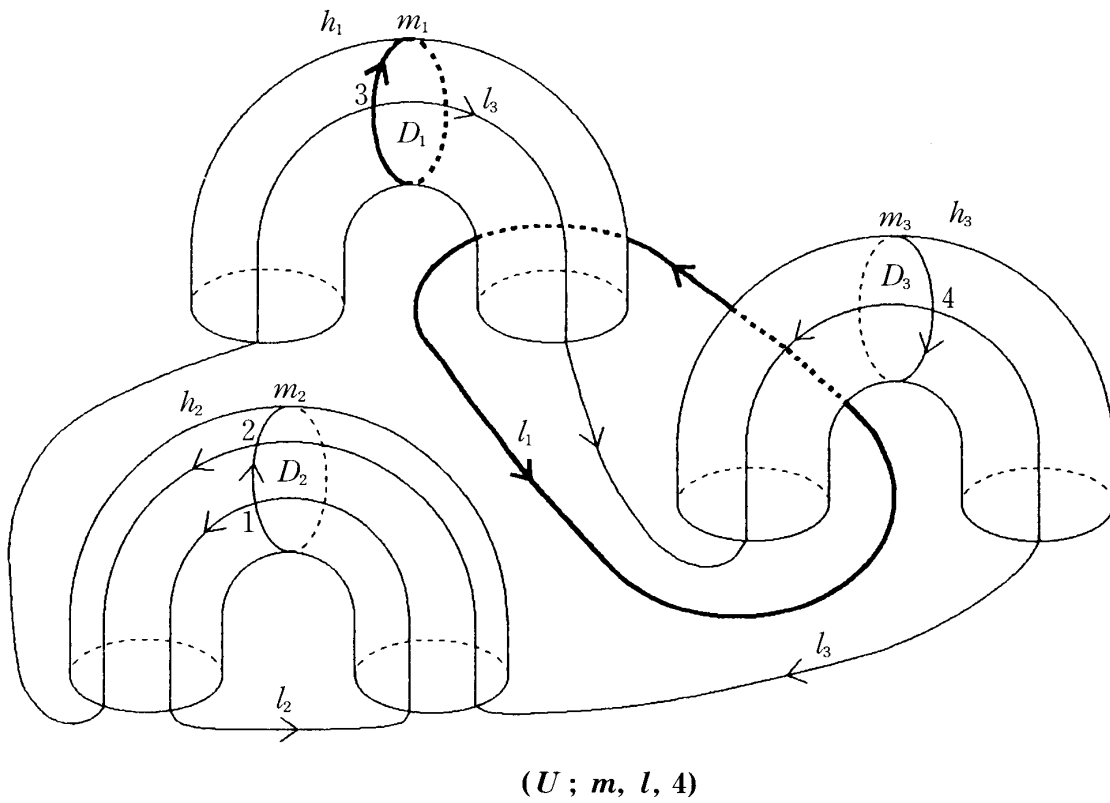
We give transformations from the disconnected H-diagrams $(U; m, l, 0) \cup (V; l, m, 0)$ into connected ones.

Tfm $(U; m, l, 0)$ and $(V; l, m, 0)$; assume the canonical genus 1 H-diagram of S^3 . If we take connected sum $(S^2 \times S^1) \# S^3$, then genus 2 disconnected H-diagrams $(U; m, l, 1) \cup (V; l, m, 1)$ of $S^2 \times S^1$ are obtained.



Tfm $(U; m, l, 1)$; add a handle h_3 to U so that a new longitude l_3 goes around the handles h_1, h_2 and h_3 . Then genus 3 H-diagram $(U; m, l, 4)$ is obtained. Here, we draw the longitude l_1 once more on the spherical surface ∂B_U^3 because of the convenience.

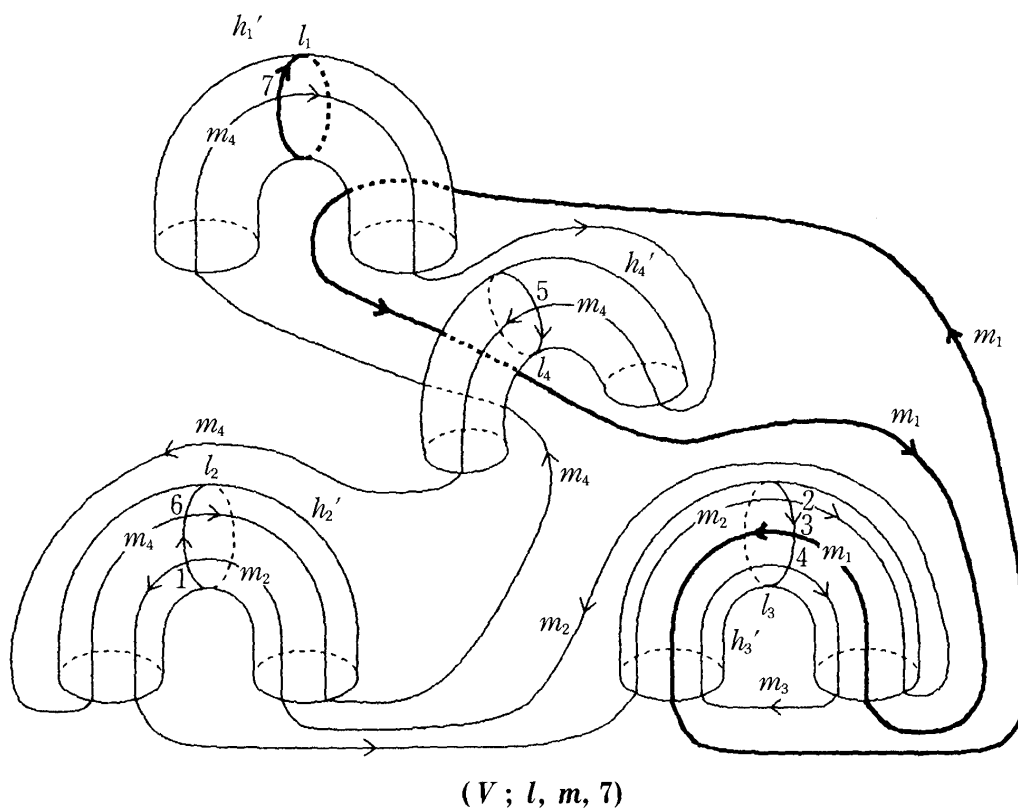
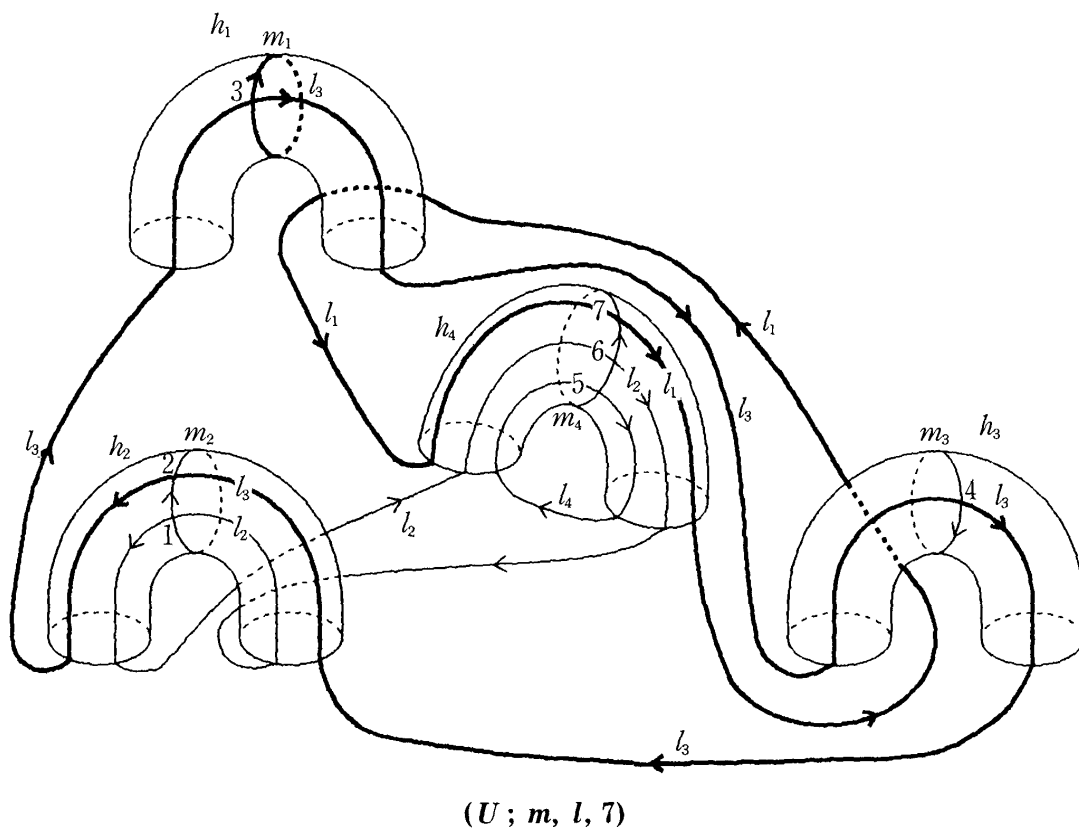
Tfm $(V; l, m, 1)$; add a handle h_3' to V and draw the longitudes m_1, m_2, m_3 so that they become the dual diagram to $(U; m, l, 4)$. Then genus 3 H-diagram $(V; l, m, 4)$ is obtained.



Tfm $(V ; l, m, 4)$; add a handle h_4' to V so that a new longitude m_4 goes around the handles h_1' , h_2' and h_4' . Then genus 4 H-diagram $(V ; l, m, 7)$ is obtained.

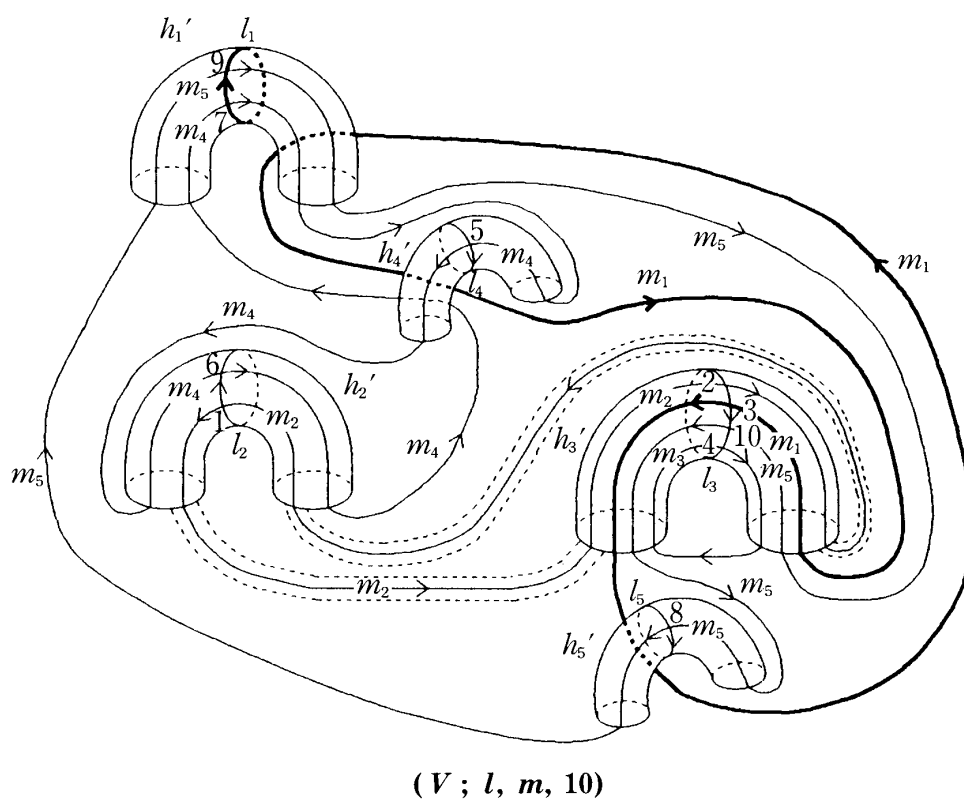
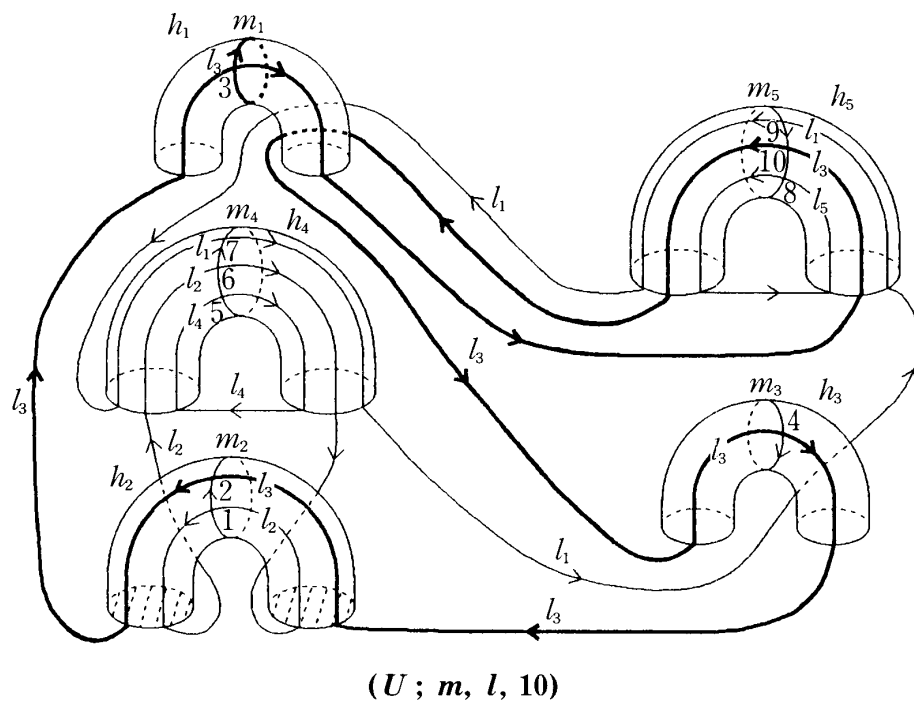
Tfm $(U ; m, l, 4)$; add a handle h_4 to U and draw the longitudes l_1 , l_2 , l_3 and l_4 so that they

become the dual diagram to $(V; l, m, 7)$. Then genus 4 H-diagram $(U; m, l, 7)$ is obtained. Fig. 1 is the H-cut-diagram of this.



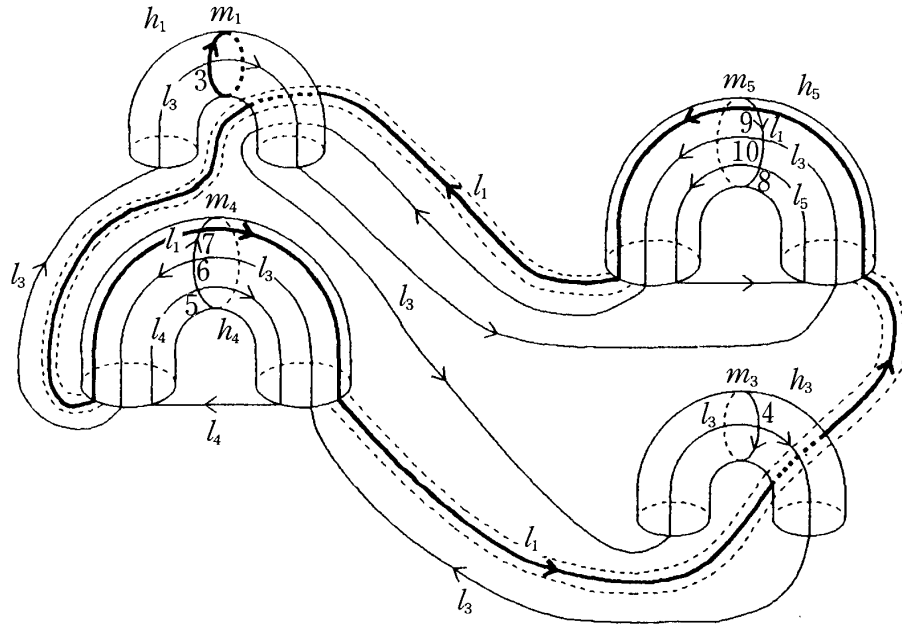
Tfm $(V; l, m, 7)$; add a handle h_5' to V so that a new longitude m_5 goes around the handles h_1' , h_3' and h_5' . Then genus 5 H-diagram $(V; l, m, 10)$ is obtained. The H-cut-diagram of this is connected.

Tfm $(U; m, l, 7)$; add a handle h_5 to U and draw the longitudes l_1, l_2, l_3, l_4 and l_5 so that they become the dual diagram to $(V; l, m, 10)$. Then genus 5 H-diagram $(U; m, l, 10)$ is obtained.

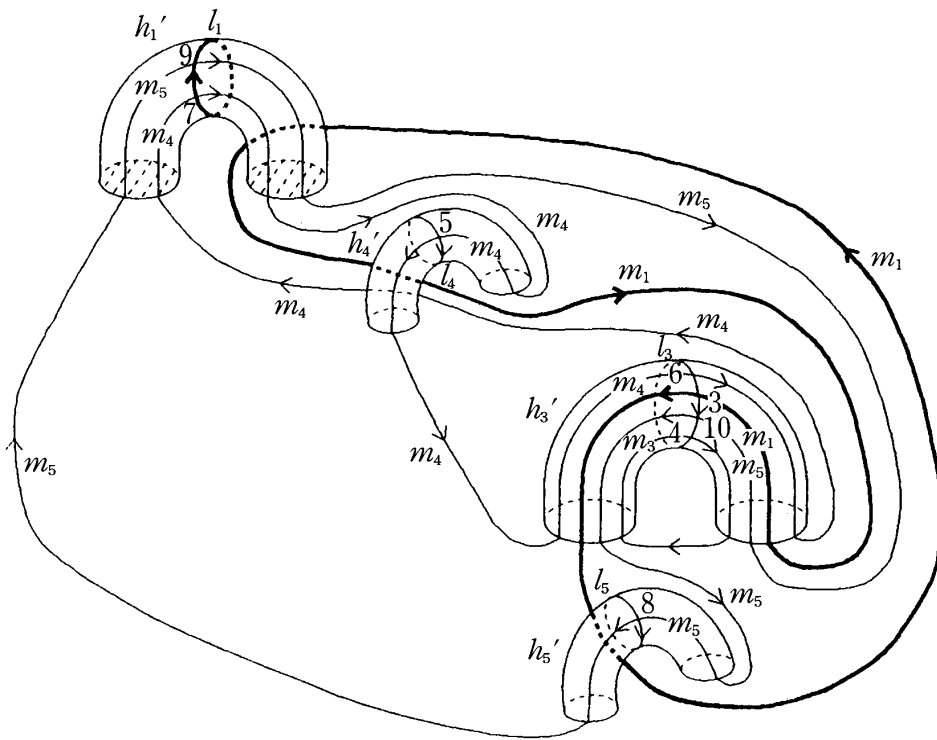


Tfm $(U ; m, l, 10)$; crushing the handle h_2 at the meridian disk $D_2(\partial D_2 = m_2)$ and cutting the deformed handle at the crushed disk (a pint), a genus 4 handlebody and a new circle is obtained from l_2 and l_3 . Next putting the label l_3 on it and reorienting it, genus 4 $(U ; m, l, 8)$ is obtained.

Tfm $(V ; l, m, 10)$; the dual genus 4 $(V ; l, m, 8)$ is obtained corresponding to the above transformation.



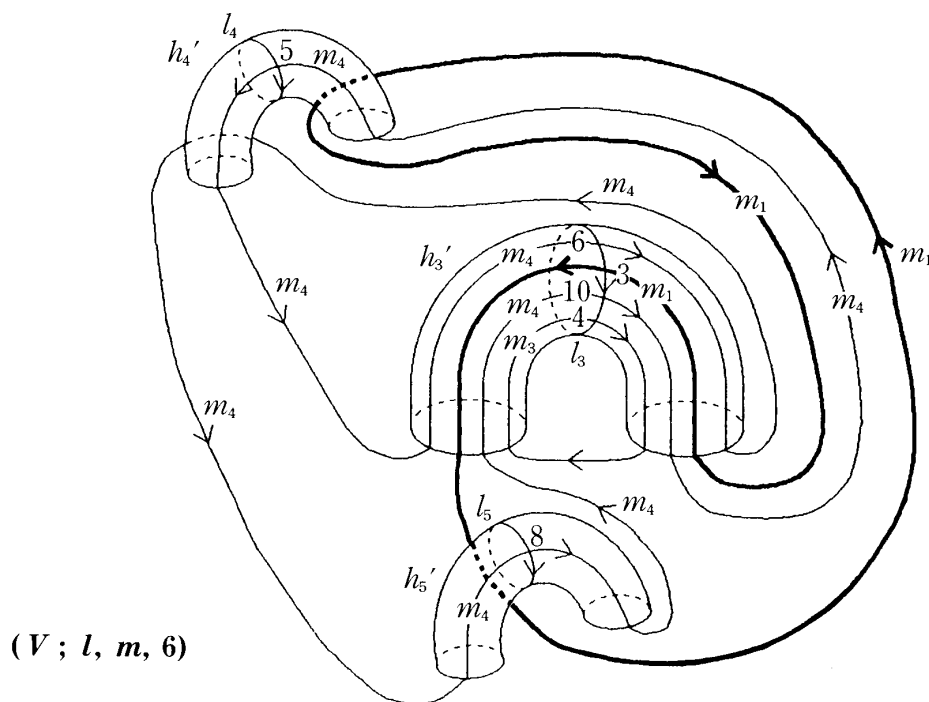
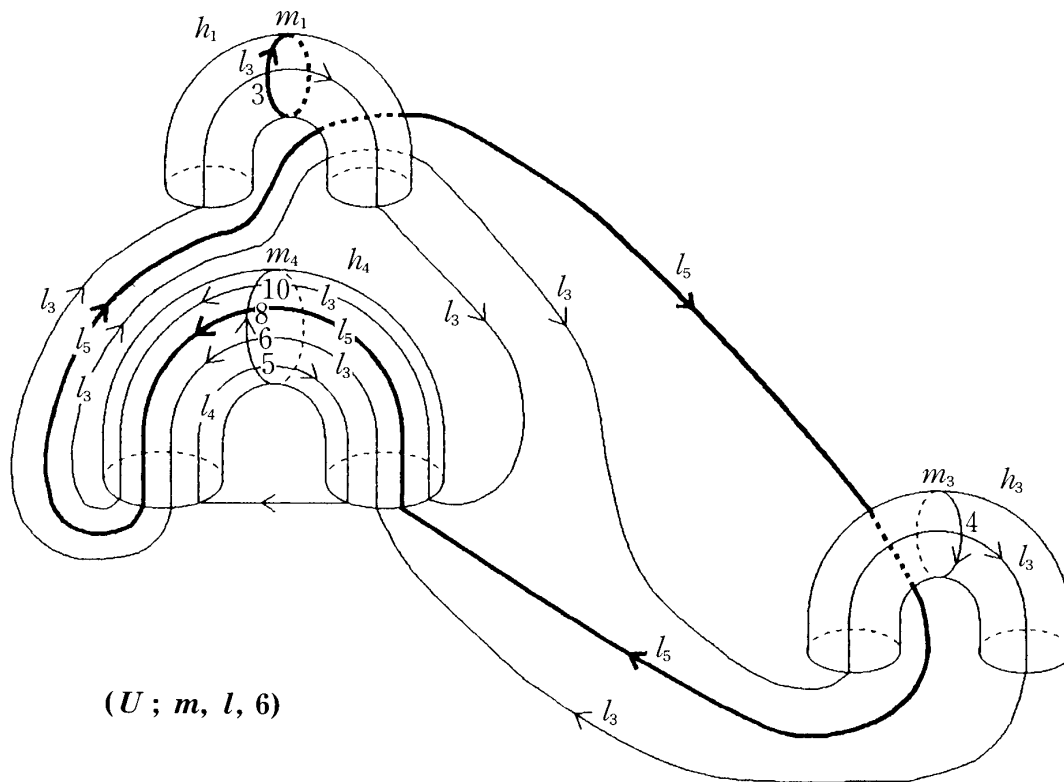
$(U ; m, l, 8)$



$(V ; l, m, 8)$

$\text{Tfm}(V; l, m, 8)$; crushing the handle h_1' at the meridian disk D_1' ($\partial D_1' = l_1$) and cutting the deformed handle at the crushed disk (a pint), a genus 3 handlebody and a new circle is obtained from m_4 and m_5 . Next putting the label m_4 on it and reorienting it, genus 3 ($V; l, m, 6$) is obtained.

$\text{Tfm}(U; m, l, 8)$; the dual genus 3 ($U; m, l, 6$) is obtained corresponding to the above transformation. The H-(cut-)diagrams of these are connected.



Example 6. Fig. 4 gives disconnected genus 2 H-diagrams $(U; m, l)$, $(V; l, m)$ of $L(7, 2) \# L(7, 4)$.

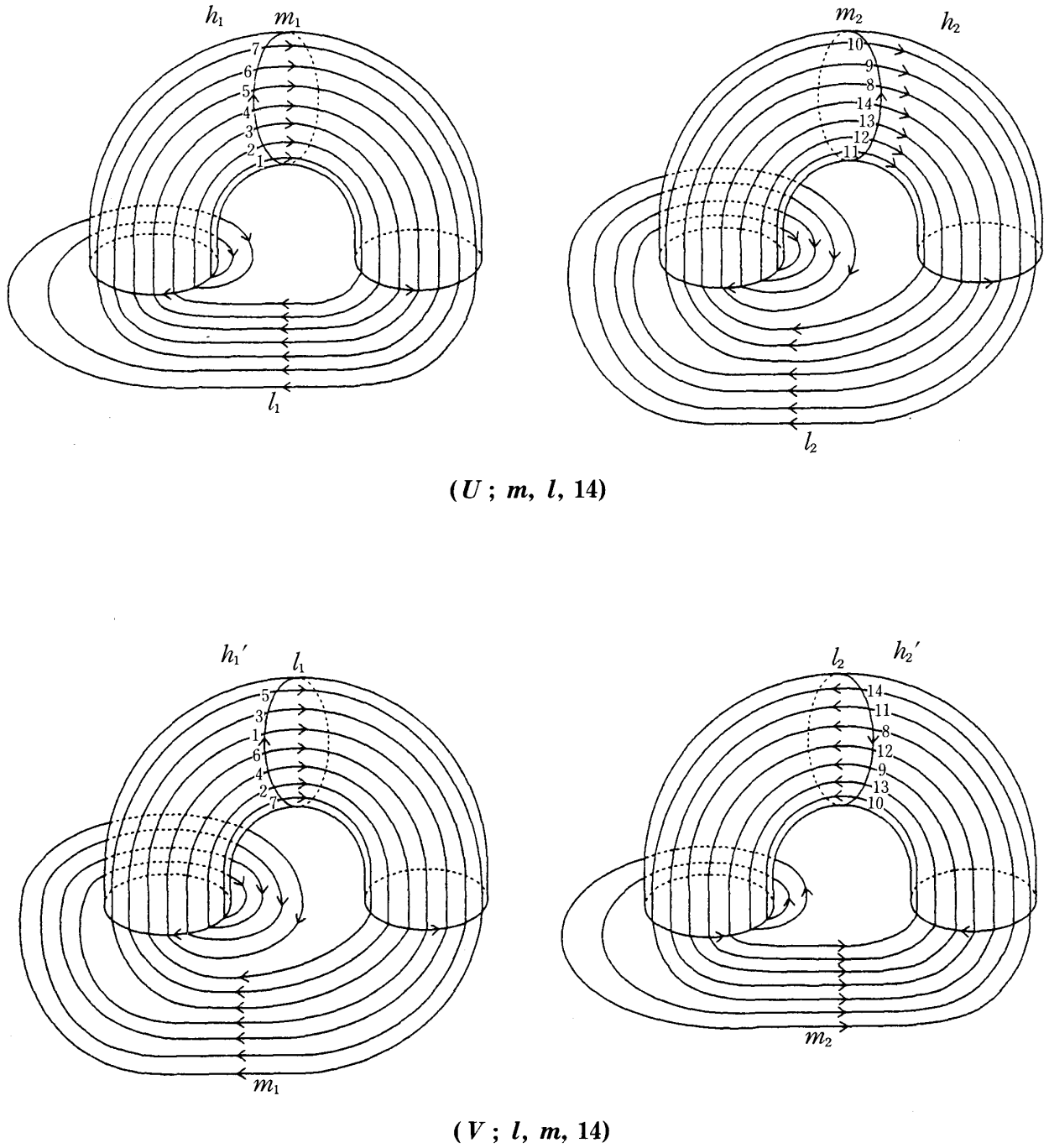
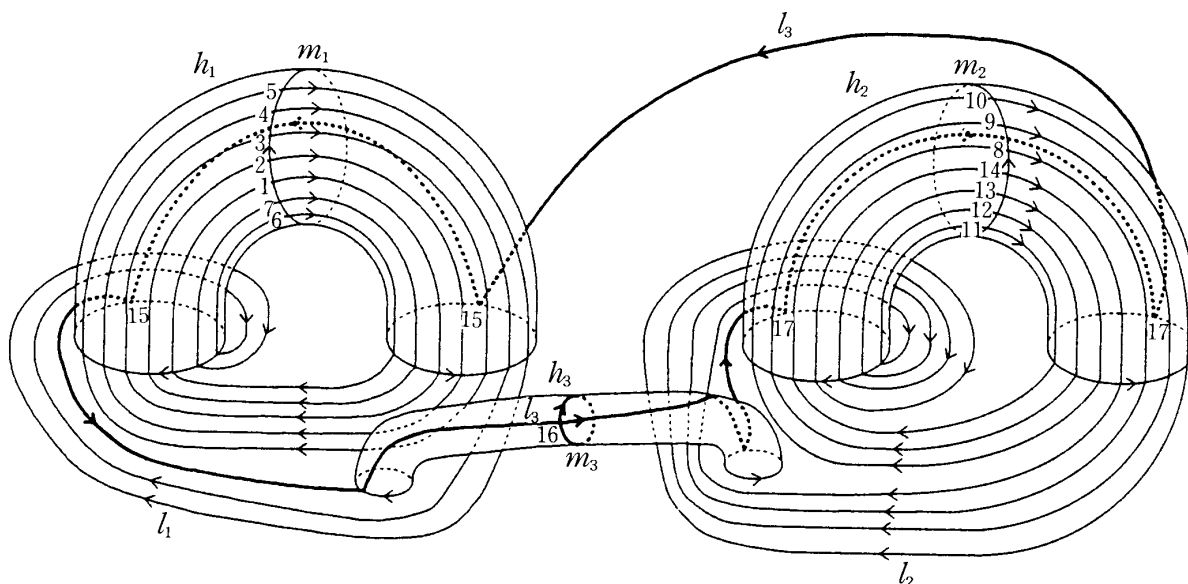


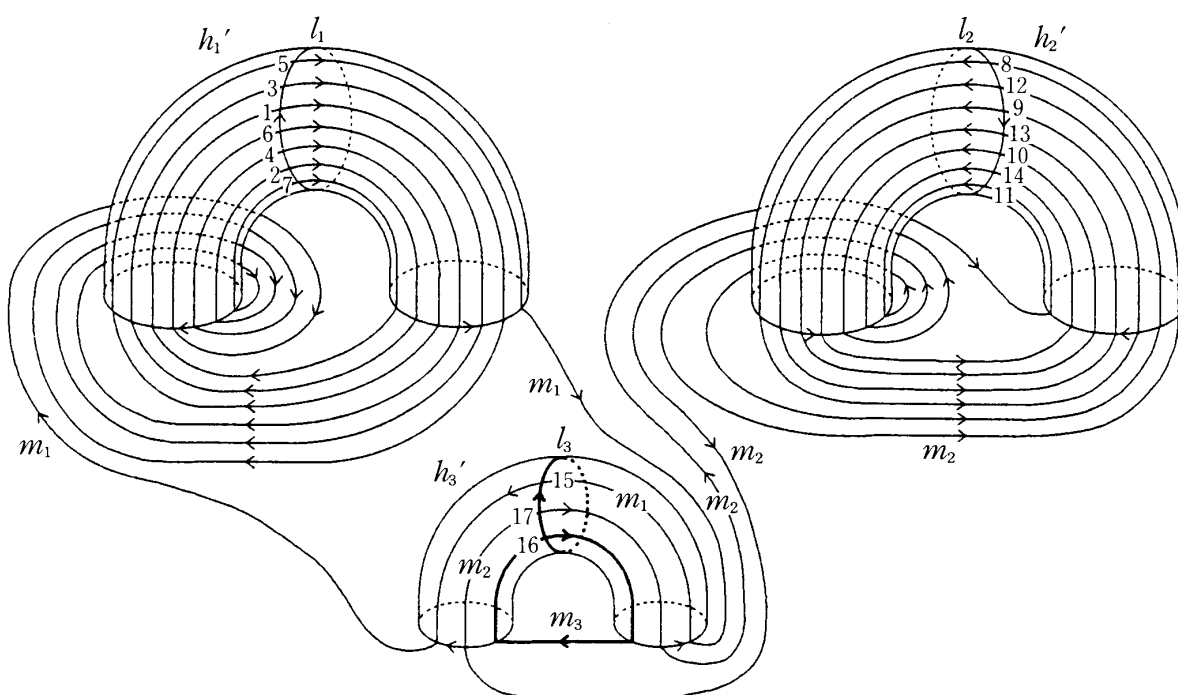
Fig. 4

Tfm $(U ; m, l, 14)$; add a handle h_3 to U so that a new longitude l_3 goes around the handles h_1, h_2 and h_3 . Then genus 3 H-diagram $(U ; m, l, 17)$ is obtained. This H-(cut-)diagram is connected.

Tfm $(V ; l, m, 14)$; the dual genus 3 $(V ; l, m, 17)$ is obtained corresponding to the above transformation.



$(U ; m, l, 17)$



$(V ; l, m, 17)$

It is very difficult to transform H-diagrams as can be seen from the above examples. Therefore, it will be desirable to transform the H-cut-diagrams by using the *DS*-deformations instead of H-diagrams. Some examples in [8] will be helpful to master the transformations of H-cut-diagrams.

Transformations from $(1 - A) \cup (1 - A')$ into $(1 - B1) \cup (1 - B1')$ and corresponding handle sliding, longitude combining and changing in [10] are very important. Henceforth we will have to move forward those studies more.

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